Quantum theory of coherence and nonlinear optics

Jan Peřina
perina@prfnw.upol.cz

Department of Optics, Palacký University, 17. listopadu 12, 771 46 Olomouc
Joint Laboratory of Optics, Palack University, 17. listopadu 50, 772 07 Olomouc, Czech Republic

We review the beginning stage of quantum theory of coherence and research performed at the Palacký University in quantum and nonlinear optics related to quantum state reconstruction, generalized superposition of signal and quantum noise, quantum Zeno effect, nonlinear optical couplers and parametric down-conversion. [DOI: 10.2971/jeos.2010.10048s]

Keywords: quantum theory of coherence, quantum optics, nonlinear optics, nonclassical light

1 INTRODUCTION

In this brief review we first describe the beginning and necessity of introducing quantum theory of coherence for dealing with low-intensity optical beams created in 1963 and give some contributions to this field obtained at the Palacký University in Olomouc in the following period. In particular we show results obtained for arbitrary multimode field operator orderings and quantum-state reconstructions on the basis of quantum moment problem. Further we introduce the so-called generalized superposition of signal and quantum noise generalizing the classical superposition of signal and noise (of coherent and chaotic fields) describing nonclassical states in terms of negative quantum noise. Attention is also devoted to the quantum Zeno effect and to nonlinear optical couplers as effective sources of nonclassical light. Strong theoretical and experimental effort directed to parametric down-conversion provided a number of points of view on nonclassical behavior of light generated in this process, including a description in terms of joint wave distributions exhibiting quantum oscillations and negative values.

2 CREATION OF QUANTUM THEORY OF COHERENCE

Classical theory of coherence was systematically developed by many authors in relation to interference patterns in classical interferometers and it is called the second-order theory because it involves the second-order correlations in field amplitudes. Such a theory started in papers by P. H. van Cittert and F. Zernike and was systematically worked out by E. Wolf (a review can be found in classical book [1] by M. Born and E. Wolf). When Hanbury Brown and Twiss extended the interferometric measurements to include intensity correlations, i.e. the fourth-order amplitude correlations, it was still possible to formulate the corresponding theory in classical terms (for a review, e.g. [2]) treating natural (chaotic) sources. However, with development of other sources than chaotic (Gaussian) ones, in particular with development of lasers, it was necessary to build a general theory of coherence, involving correlations of all orders, which can uniquely be determined from the second-order correlations for Gaussian fields only. This was done in about 1964–1969 [3]–[5]]. However, any application of this theory, mostly in optical imaging, needs strong optical fields in which quantum noise plays no role. When weak quantum fields are involved, quantum electrodynamics must be applied. This was done for the first time by R. J. Glauber in 1963 [6, 7] (Nobel Price in 2005) adopting the so-called coherent states $|\alpha\rangle$ introduced by E. Schrödinger in 1927 [8]. These states represent some opposite states to Fock (number) states (definite photon numbers, uncertain phase), describing fields with fluctuating photon numbers and definite complex amplitudes, which is necessary for description of coherence as a cooperative boson phenomenon. This is just the beginning of the quantum theory of coherence. The most important step in this development was to express the density matrix in terms of the coherent states,

$$\hat{\rho} = \int P(\alpha)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha, \tag{1}$$

which represents a superposition of density matrices for coherent states weighted by the function $P(\alpha)$; the integral is taken over the complex plane. It is now called the Glauber-Sudarshan diagonal representation of the density matrix. For natural chaotic fields this function is Gaussian and both the classical and quantum descriptions are fully equivalent mathematically and physically as well. Using this representation also all quantum correlation functions measured by photodetectors (normally ordered-all creation operators are to the left of all annihilation operators) can formally be expressed in a "classical" form, which led to a formulation of the so-called equivalence theorem. This caused a discussion especially among L. Mandel and E. Wolf and R. J. Glauber about the meaning of this theorem (see also discussions in [3]). It was successively clear that this equivalence is generally only formal and it is valid only if the weighting function P behaves as classical distribution function (it is regular and non-negative). However, in general it is expressed in terms of the Fock density matrix elements as double series of derivatives of the δ

Received January 14, 2010; published September 23, 2009 ISSN 1990-2573

function, it can take on negative values and be very singular (it is the so-called generalized function-ultradistribution) [9]. Then a formal mathematical equivalence does not mean the physical equivalence and such unusual mathematical properties of distribution functions reflect just quantum physical properties of optical fields having no classical analogue. This opened a large field for applications of the process of parametric frequency down-conversion to theoretical and experimental studies of quantum properties of light, development of quantum homodyne tomography and reconstructions of quantum states and creation of new directions in quantum transmission, processing and recording information. Owing to efforts of many working places over the world the methods of quantum optics were applied for proving the validity of quantum principles, for description of propagation of light in random and nonlinear media, and for generating nonclassical fields, in particular in nonlinear optical processes, useful in practical applications, such as quantum cryptography, very precise quantum measurements (for example in efforts to detect gravitational waves), quantum information, quantum teleportation, quantum computing, etc. One of the first monographs giving the classical and quantum theory of coherence and their relations was published in 1972 [2].

It should be mentioned that the first summer school devoted to creation of quantum optics was organized in 1969 by R. M. Sillito and P. Farago at Carberry Tower close to Edinburgh [10]. It was possible to meet many important people in this field there: R. J. Glauber, A. Kastler, N. Bloembergen, A. L. Shawlow, H. Haken, T. W. B. Kibble, W. H. Louisell, E. R. Pike, G. W. Series, G. Toraldo di Francia, L. Mandel, D. F. Walls, F. Haake, R. Graham, and T. Hänsch among others.

3 ORDERING OF FIELD OPERATORS AND RECONSTRUCTION OF QUANTUM STATES

Fundamental works in this field were published by Agarwal and Wolf [11, 12, 13] and Cahill and Glauber [14, 15]. We discuss here the nontrivial multimode generalization leading to the generalized photodetection equation and interesting inverse problem for determination of wave properties of radiation from photocount measurements [2] (Chapter 16 and references therein).

Describing an M-mode field by the number operator $\hat{n} = \sum_\lambda \hat{a}_\lambda^\dagger \hat{a}_\lambda$ and the corresponding wave integrated intensity $W = \sum_\lambda |\alpha_\lambda|^2$ (α_λ being the complex amplitude in the mode λ corresponding to the photon annihilation operator $\hat{a}_\lambda$), we can derive the following relation between the s-ordered and number generating functions

$$\langle \exp(\mathrm{i}y\hat{n})\rangle_s = [1-(1-s)\mathrm{i}y/2)]^{-M} \left\langle \left[\frac{1+(1+s)\mathrm{i}y/2}{1-(1-s)\mathrm{i}y/2}\right]^n \right\rangle, \tag{2}$$

where iy is a parameter of the generating function, or we can obtain the relation between s_1- and s_2-ordered generating functions (s equals 1, 0 and -1 for normal, symmetric and antinormal orderings, respectively)

$$\langle \exp(\mathrm{i}y\hat{n})\rangle_{s_2} = [1+(s_2-s_1)\mathrm{i}y/2]^{-M} \times \left\langle \exp\left[\frac{\mathrm{i}y\hat{n}}{1+(s_2-s_1)\mathrm{i}y/2}\right]\right\rangle_{s_1}. \tag{3}$$

In particular from Eq. (2) we can derive the generalized photodetection equation [16, 17] relating the photon number distribution $p(n)$ and quasidistribution $P(W,s)$ of the integrated intensity W related to s-ordering

$$p(n) = \frac{1}{\Gamma(n+M)}\left(\frac{2}{1+s}\right)^M \left(\frac{s-1}{s+1}\right)^n \times \int_0^\infty P(W,s) L_n^{M-1}\left(\frac{4W}{1-s^2}\right) \exp\left(-\frac{2W}{1+s}\right) \mathrm{d}W, \tag{4}$$

where L_n^{M-1} is the Laguerre polynomial (in the Morse-Feshbach normalization) and Γ is the Gamma function. In the limit $s \to 1$ for the normal operator ordering related to photodetection, we obtain the standard Mandel photodetection equation as the average of the Poisson kernel. Excepting a number of particular cases, we can give general relations between integrated intensities for two orderings and for their moments

$$P(W,s_2) = \frac{2}{s_1-s_2}\int_0^\infty \left(\frac{W}{W'}\right)^{\frac{(M-1)}{2}} \exp\left[-\frac{2(W+W')}{s_1-s_2}\right] \times I_{M-1}\left(4\frac{(WW')^{1/2}}{s_1-s_2}\right) P(W',s_1)\mathrm{d}W', \tag{5}$$

where $\mathrm{Re}[s_1] > \mathrm{Re}[s_2]$, I_{M-1} is the modified Bessel function, and

$$\langle W^k\rangle_{s_2} = \frac{k!}{\Gamma(k+M)}\left(\frac{s_1-s_2}{2}\right)^k \left\langle L_k^{M-1}\left(\frac{2W}{s_2-s_1}\right)\right\rangle_{s_1}. \tag{6}$$

It is worth noting that this formulation provides a basis for solving various inverse problems, e.g. to determine $P(W,s)$ from the photocount measurements giving $p(n)$. We can obtain, e.g., for normal ordering

$$P_\mathcal{N}(W) = \exp[-(\zeta-1)W]\sum_{j=0}^\infty c_j L_j^0(\zeta W), \tag{7}$$

where

$$c_j = \zeta \sum_{s=0}^j p(s)\frac{(-\zeta)^s}{(j-s)!\,s!} \tag{8}$$

or more generally for M-mode systems

$$P_\mathcal{N}(W) = W^{M-1}\sum_{j=0}^\infty c_j L_j^{M-1}(W), \tag{9}$$

where

$$c_j = \frac{j!}{\Gamma(j+M)}\sum_{s=0}^j \frac{(-1)^s p(s)}{(j-s)!\,\Gamma(s+M)}. \tag{10}$$

These reconstruction formulae have the advantage that they can be used to reconstruct approximate quasidistributions from a finite number of measurements and also they provide some regularized forms of quasidistributions (when also phase should be included, a basis $\{\alpha^\lambda L_n^\lambda(|\alpha|^2)\}$ for a field complex amplitude α can be used, which is orthogonal with the weight $\exp(-|\alpha|^2)$).

The above formulae are able to reconstruct also the nonclassical behavior of the quasidistribution, e.g. its negative values, as applied, for instance, by Klyshko [18]. This can be done by defining a set of "moments" $M_n = p(n)n!$ and quadratic forms

$$q_m = \sum_{j,k=0}^{m} M_{j+k} u_j u_k \tag{11}$$

for an arbitrary nontrivial real vector $\{u_j\}$ and $m = 0, 1, \ldots$. If q_m are larger than zero for every m, then it is possible to construct the non-negative distribution $P_{\mathcal{N}}(W)$ from a given sequence of moments M_n in a unique way. However, if for some m this quadratic form equals zero, there exists a unique quasidistribution the support of which is composed of a finite number of points equal to the minimal number of the m's for which the form equals zero. This criterion can be expressed in terms of determinants $\mathcal{D}_m = \mathrm{Det}[M_{j+m}], j = 0, 1, \ldots, m$, because if these determinants are positive, then also the forms are positive [19] (Section 3.8).

4 GENERALIZED SUPERPOSITION OF SIGNAL AND QUANTUM NOISE

The above formalism can be applied for obtaining multimode superposition of coherent signal and noise, giving a generalization of the well-known Mandel-Rice formula and permitting a quantum generalization. Assuming a rectangular broad-band spectrum of noise, the M-mode (temporal, spatial and polarization) generating function is the generating function for the Laguerre polynomials L_n^{M-1} (a polarized superposition is assumed),

$$\langle \exp(\mathrm{i}sW)\rangle_{\mathcal{N}} = \left(1 - \mathrm{i}s\frac{\langle n_{\mathrm{ch}}\rangle}{M}\right)^{-M} \exp\left[\frac{\mathrm{i}s\langle n_{\mathrm{c}}\rangle}{1 - \mathrm{i}s\frac{\langle n_{\mathrm{ch}}\rangle}{M}}\right], \tag{12}$$

where is is the parameter of the generating function, and $\langle n_{\mathrm{c}}\rangle$ and $\langle n_{\mathrm{ch}}\rangle$ are the mean number of coherent signal and the mean number of chaotic noise photons, respectively [19] (Section 5.3 and references therein).

The corresponding distribution of the integrated intensity W is expressed in terms of modified Bessel functions,

$$\begin{aligned} P_{\mathcal{N}}(W) &= \frac{M}{\langle n_{\mathrm{ch}}\rangle}\left(\frac{W}{\langle n_{\mathrm{c}}\rangle}\right)^{(M-1)/2} \exp\left(-\frac{W + \langle n_{\mathrm{c}}\rangle}{\langle n_{\mathrm{ch}}\rangle}M\right) \\ &\times I_{M-1}\left(2M\frac{[\langle n_{\mathrm{c}}\rangle W]^{1/2}}{\langle n_{\mathrm{ch}}\rangle}\right), \quad \text{for } W \geq 0, \qquad (13a) \\ &= 0, \quad \text{for } W < 0. \qquad (13b) \end{aligned}$$

For the photon-number distribution we have

$$\begin{aligned} p(n) &= \frac{1}{\Gamma(n+M)}\left(1 + \frac{\langle n_{\mathrm{ch}}\rangle}{M}\right)^{-M}\left(1 + \frac{M}{\langle n_{\mathrm{ch}}\rangle}\right)^{-n} \\ &\times \exp\left(-\frac{\langle n_{\mathrm{c}}\rangle M}{\langle n_{\mathrm{ch}}\rangle + M}\right) \\ &\times L_n^{M-1}\left(-\frac{\langle n_{\mathrm{c}}\rangle M^2}{\langle n_{\mathrm{ch}}\rangle(\langle n_{\mathrm{ch}}\rangle + M)}\right). \end{aligned} \tag{14}$$

The corresponding factorial moments are

$$\langle W^k\rangle_{\mathcal{N}} = \frac{k!}{\Gamma(k+M)}\left(\frac{\langle n_{\mathrm{ch}}\rangle}{M}\right)^k L_k^{M-1}\left(-\frac{\langle n_{\mathrm{c}}\rangle M}{\langle n_{\mathrm{ch}}\rangle}\right). \tag{15}$$

In special cases we obtain expressions for fully coherent or fully chaotic light, e.g. Poisson or Mandel-Rice distributions.

For nonlinear two photon processes (sub-harmonic generation) these formulae can be generalized so that they can describe nonclassical effects. The generating function "splits" as follows

$$\begin{aligned} \langle \exp(\mathrm{i}sW)\rangle_{\mathcal{N}} &= [1 - \mathrm{i}s(E-1)]^{-M/2}[1 - \mathrm{i}s(F-1)]^{-M/2} \\ &\times \exp\left[\frac{\mathrm{i}sA_1}{1 - \mathrm{i}s(E-1)} + \frac{\mathrm{i}sA_2}{1 - \mathrm{i}s(F-1)}\right], \end{aligned} \tag{16}$$

where $E = B - |C|$, $F = B + |C|$ represent quantum noise, $B = \langle \Delta\hat{a}^\dagger \Delta\hat{a}\rangle$ and $C = \langle (\Delta\hat{a})^2\rangle$ represent phase independent and phase dependent amplitude fluctuations, respectively, number 1 subtracted from E, or F represents the subtraction of vacuum fluctuations in the normal operator ordering and $A_{1,2}$ are signal components

$$A_{1,2} = \frac{1}{2}\left[\sum_{j=1}^{M}|\xi_j(t)|^2 \mp \frac{1}{2|C|}\left(C^*\sum_{j=1}^{M}\xi_j^2(t) + \mathrm{c.c.}\right)\right], \tag{17}$$

where $\xi_j(t)$ are time developed complex amplitudes and c.c. denotes the complex conjugate terms. For example, for the degenerate optical parametric process with strong classical coherent pumping we obtain

$$B = \cosh^2(gt), \tag{18a}$$
$$C = \frac{\mathrm{i}}{2}\sinh(2gt)\exp(\mathrm{i}\phi), \tag{18b}$$
$$E - 1 = \frac{1}{2}[\exp(-2gt) - 1] \leq 0, \tag{18c}$$
$$F - 1 = \frac{1}{2}[\exp(2gt) - 1] \geq 0, \tag{18d}$$
$$A_{1,2} = \frac{1}{2}|\xi|^2\exp(\mp 2gt)[1 \mp \sin(2\theta - \phi)] \geq 0, \tag{18e}$$

where g is a nonlinear coupling constant proportional to the quadratic susceptibility and pumping real amplitude, ϕ is the pump phase and θ is the initial phase of the signal. We see that the quantum noise component $E-1$ is negative for all times and we have a superposition of a signal with negative quantum noise. If the initial phases are suitably related, e.g. $2\theta - \phi = -\pi/2$, the first factor in the generating function (Eq. (16)) involving A_1 and $E-1$ is dominating and the photon distribution reduces to sub-Poissonian distribution for short times (the normal wave variance $\langle(\Delta W)^2\rangle_{\mathcal{N}}$ is negative). For later times oscillations in photon number distribution $p(n)$ occur having no classical analogue. The photon-number distribution and its factorial moments are expressed in terms of the Laguerre polynomials as follows

$$\begin{aligned} p(n) &= \frac{1}{(EF)^{M/2}}\left(1 - \frac{1}{F}\right)^n \exp\left(-\frac{A_1}{E} - \frac{A_2}{F}\right) \\ &\times \sum_{k=0}^{n}\frac{1}{\Gamma(k+M/2)\Gamma(n-k+M/2)}\left(\frac{1-1/E}{1-1/F}\right)^k \\ &\times L_k^{M/2-1}\left(-\frac{A_1}{E(E-1)}\right)L_{n-k}^{M/2-1}\left(-\frac{A_2}{F(F-1)}\right), \end{aligned} \tag{19}$$

$$\langle W^k\rangle = k!(F-1)^k \sum_{l=0}^{k} \frac{1}{\Gamma(l+M/2)\Gamma(k-l+M/2)} \left(\frac{E-1}{F-1}\right)^l \times L_l^{M/2-1}\left(-\frac{A_1}{E-1}\right) L_{k-l}^{M/2-1}\left(-\frac{A_2}{F-1}\right). \quad (20)$$

Denoting $p_1(n)$ and $p_2(n)$ as partial distributions related to factors containing A_1, E and A_2, F, respectively, the resulting distribution is a discrete convolution,

$$p(n) = \sum_{k=0}^{n} p_1(n-k)p_2(k). \quad (21)$$

In the above case of phase relations, the partial distribution p_1 is oscillating and takes on negative values whereas the partial distribution p_2 is geometric and nonnegative. As a result of the discrete convolution of Eq. (21), a nonnegative photon-number distribution arises with quantum oscillations.

Thus we have a generalization of the classical superposition of signal and noise extending it to negative quantum noise components, which makes it possible to include nonclassical effects. For difference-number distributions further generalizations including complex noise are possible (see, e.g., [33]).

5 QUANTUM ZENO EFFECT

The quantum Zeno effect refers to the inhibition of the isolated temporal evolution of a dynamical system when the observation of such evolution is attempted (see a review [20]). This observation is usually described by frequent measurements on the system performed in order to discover whether the initial system has changed or not. In the limit of very frequent measurements, continuous observation, or arbitrary high resolution, it may happen that the system is locked on its initial state and the evolution, which is the aim of the observation, is in fact inhibited and does not occur. The effect was demonstrated using atomic transitions, neutron spin dynamics, etc. We have demonstrated it using nonlinear effects of parametric down-conversion and Raman scattering [21]–[23].

We can consider a nonlinear crystal of length L which is pumped by a strong, classical and coherent field to produce pairs of signal and idler photons via spontaneous parametric down-conversion. Using the interaction picture, this interaction is described by the effective Hamiltonian

$$\hat{H} = \hbar g(\hat{a}_s^\dagger \hat{a}_i^\dagger + \hat{a}_s \hat{a}_i), \quad (22)$$

where $\hat{a}_s$ and $\hat{a}_i$ are the slowly varying annihilation operators for signal and idler beams respectively, and g is a coupling parameter depending on the pump field and the quadratic susceptibility of the medium. We have also assumed the frequency resonance condition $\omega_p = \omega_s + \omega_i$, where ω_p, ω_s and ω_i are the frequencies of the pump, signal and idler beams, respectively. We will denote by τ the interaction time associated with the length L of the crystal. We focus on the generation of the signal from the vacuum. The interaction Hamiltonian, together with the standard quantum lossy mechanism [19], produce after the interaction time τ the following general relation between the output operator $\hat{a}_s'$ for the signal field and the input signal and idler operators $\hat{a}_s$ and $\hat{a}_i$:

$$\hat{a}_s' = \mu\hat{a}_s + \nu\hat{a}_i^\dagger + \hat{L}_s, \quad (23)$$

where

$$\mu = \exp(-\gamma_s\tau/2)\cosh(g\tau), \quad (24a)$$
$$\nu = -\mathrm{i}\exp(-\gamma_s\tau/2)\sinh(g\tau), \quad (24b)$$

$$\hat{L}_s = \sum_l w_{sl}\hat{b}_{sl} \quad (25)$$

is an operator related to the Langevin force for signal losses and it holds that

$$\sum_l |w_{sl}|^2 = 1 - \exp(-\gamma_s\tau); \quad (26)$$

here $\hat{b}_{sl}$ are initial reservoir operators, γ_s is a signal damping coefficient and we have neglected rotating reservoir terms, which give negligible contribution in the optical region.

Now we assume that the crystal is divided into N equal parts of length $\Delta L = L/N$, with associated interaction time $\Delta\tau = \tau/N$ within each part. We can assume that the signal beams of each part are perfectly superimposed and aligned, and that reflection at each piece can be avoided or made negligible, for instance, embedding the N pieces in a linear medium with the same refractive index. On the other hand, the idler path is interrupted after each piece by means of mirrors, for instance. The output idler beams after each piece are removed from the idler path being replaced by new idler beams which are in vacuum. This modification makes it possible to observe the N output idler beams to detect the emission when it occurs, for instance, by means of N photodetectors. Then, the moment of emission can be inferred with accuracy $\Delta\tau$, and the relative resolution is given by the number of pieces $\Delta\tau/\tau = 1/N$. All losses related to various imperfections can be included in the lossy reservoirs.

Now we can examine the influence of this arrangement on the single-photon emission. The signal output operator after N pieces reads

$$\hat{a}_{sN}' = \mu^N\hat{a}_{s1} + \sum_{k=1}^{N} \mu^{N-k}\nu\hat{a}_{ik}^\dagger + \frac{1-\mu^N}{1-\mu}\hat{L}_{sN}, \quad (27)$$

where the coefficients (24) are considered for $\Delta\tau$. Now the probability to have one signal photon is given by

$$\langle \hat{a}_{sN}'^\dagger \hat{a}_{sN}'\rangle_{\mathrm{vac}} = N(g\Delta\tau)^2 + N^2\langle n_{rsN}\rangle\gamma_s\Delta\tau \quad (28a)$$
$$= \frac{(g\tau)^2}{N} + \langle n_{rs}\rangle\gamma_s\tau, \quad (28b)$$

where we have considered $g\Delta\tau \ll 1, \gamma_{s,i}\Delta\tau \ll 1$ so that $\mu \simeq 1, \nu \simeq -\mathrm{i}g\Delta\tau$, $\sum_l |w_{sl}|^2 \simeq \gamma_s\Delta\tau$, $\langle n_{rs}\rangle$ are mean numbers of reservoir oscillators, which are negligibly small in the optical region for room temperatures, and also signal beam is initially in vacuum. We have two terms here, the first one arises from nonlinear dynamics, the second one from the signal lossy mechanism. The first term exhibits the quantum Zeno effect because no signal photons are radiated if the accuracy of the observation is increased by increasing N. In the limit of N tending to infinity the probability of signal photon emission tends to zero and there is no emission at all. We can note that whether the attempted measurement on the idler modes is actually made or not appears to make no difference. It is sufficient that it could be made. In general the losses in the signal beam degrade the quantum Zeno effect. This lossy effect is nonlocal because the measurement on the idler beam reduces the Zeno effect in the signal beam through its losses.

We see that for unobserved system the N emitters are stimulated by the same vacuum, imparting phase correlations between them. On the observed system the pieces are influenced by different and statistically independent vacuum fields leading to mutually incoherent emissions. This refers rather to the idler beam. However, through the strong quantum nonlocal correlations also signal beam is controlled. Alternatively, the probability of emission on the unobserved system can be considered as the constructive interference between N possible and intrinsically indistinguishable ways for the emission to occur. When we interrupt the idler path N times, these ways become distinguishable by the possible detection of the idler photon. This possibility wipes out the interference, and the emission is modified. In the optical region for room temperatures the obtained lossy effects in the signal beam are not critical for observation, $N_{\max}$ being of about 10^{20}.

If also phase mismatch in the nonlinear process is taken into account, the signal photon emission can be supported and the anti-Zeno effect may arise [22, 24] in which signal photon emission is increased by the measurement. The above arrangement can be modified using Kerr effect [21]. The influence of losses is similar.

6 NONLINEAR OPTICAL COUPLERS

The above results were continued by systematic studies of propagation of radiation in random and nonlinear media [25] and of evolution of quantum statistics in nonlinear optical processes [19, 26, 27]. Much effort has been devoted to study quantum statistical properties of nonlinear optical couplers composed of two or three linear and nonlinear waveguides connected by evanescent waves [28, 29] used for generation and propagation of nonclassical light. Nonlinear waveguides operating by degenerate as well as nondegenerate optical parametric processes, by Raman scattering or Brillouin scattering and Kerr effect, and also a bandgap coupler were considered. Squeezing of vacuum fluctuations, sub-Poissonian photon behavior, collapses and revivals of quantum oscillations and properties of quantum phase were examined in single and compound modes in fully quantum way in short-length approximations or in parametric approximation of strong coherent pumping. In some cases symbolic computations to obtain higher-order fully quantum solutions were applied. Both regimes of codirectional and contradirectional propagation were considered. Also substituting schemes and stability problems were investigated.

Such composed nonlinear devices can be applied not only as sources for generation and propagation of light exhibiting nonclassical properties and as switching devices, but also as elements for quantum measurements using linear waveguide as a continuous probe device [30] and for investigation of quantum coherence [31]. The most comprehensive review of aspects of quantum propagation of light beams can be found in a publication [32].

7 JOINT DISTRIBUTIONS IN OPTICAL PARAMETRIC DOWN-CONVERSION

We begin with the standard two-mode description of parametric down-conversion described by the interaction Hamiltonian

$$\hat{H}_{\text{int}} = -\hbar g \hat{a}_1 \hat{a}_2 \exp(\mathrm{i}\omega t - \mathrm{i}\phi) + \text{h.c.}, \tag{29}$$

where $\hat{a}_1$ ($\hat{a}_1^\dagger$) and $\hat{a}_2$ ($\hat{a}_2^\dagger$) represent annihilation (creation) operators of signal and idler beams, respectively, g is a real coupling constant proportional to the quadratic susceptibility $\chi^{(2)}$ and the real amplitude of pumping $|\xi|$, t is interaction time, $\omega = \omega_1 + \omega_2$ is the pumping frequency, ω_1 and ω_2 are the signal and idler frequencies, respectively, ϕ is the pumping phase and h.c. means the Hermitian conjugate term. The well-known solutions in the interaction picture are

$$\hat{a}_1(t) = \hat{a}_1(0)u(t) + \mathrm{i}\hat{a}_2^\dagger(0)v(t)\exp(\mathrm{i}\phi), \tag{30a}$$
$$\hat{a}_2(t) = \hat{a}_2(0)u(t) + \mathrm{i}\hat{a}_1^\dagger(0)v(t)\exp(\mathrm{i}\phi), \tag{30b}$$

where $u(t) = \cosh(gt)$, $v(t) = \sinh(gt)$ and functions $\exp(-\mathrm{i}\omega_j t)$, $j = 1, 2$ are omitted in the interaction picture. Losses and noise are usually described in the standard quantum consistent way assuming that both the modes are coupled to large Gaussian-Markovian reservoir systems (see, e.g., [19], Chapter 7). In this case the field amplitudes are damped (losses) and additional contributions are obtained from reservoir Langevin forces (noise). Now we can use the normal quantum characteristic function

$$C_{\mathcal{N}}(\beta_1, \beta_2, t) = \mathrm{Tr}\Big\{\hat{\rho}\exp(\beta_1\hat{a}_1^\dagger(t) + \beta_2\hat{a}_2^\dagger(t)) \times \exp(-\beta_1^*\hat{a}_1(t) - \beta_2^*\hat{a}_2(t))\Big\}, \tag{31}$$

which is obtained, using the above solutions, in the Gaussian form [19]

$$C_{\mathcal{N}}(\beta_1, \beta_2) = \exp\Big[-|\beta_1|^2 B_1 - |\beta_2|^2 B_2 + D_{12}\beta_1^*\beta_2^* + D_{12}^*\beta_1\beta_2\Big] \tag{32}$$

where β_1, β_2 are parameters of the characteristic function; we assume the spontaneous process without loss of generality [33]. In this ideal case the quantum noise functions are $B_1 = \langle \Delta\hat{a}_1^\dagger \Delta\hat{a}_1 \rangle = B_2 = \langle \Delta\hat{a}_2^\dagger \Delta\hat{a}_2 \rangle = B = \sinh^2(gt)$, $D_{12} = \langle \Delta\hat{a}_1 \Delta\hat{a}_2 \rangle = (\mathrm{i}/2)\sinh(2gt)\exp(\mathrm{i}\phi)$. Hence the squeeze parameter is $r = gt$. When determining the Glauber-Sudarshan quasidistribution as a Fourier transform of the corresponding characteristic function [19, 26, 34]–[36], the determinant $K = B^2 - |D_{12}|^2 = -\sinh^2(gt) < 0$ is involved in the calculation and its negative value causes that this quasidistribution does not exist for all times and consequently the signal and idler beams are entangled for all times (see a discussion below Eq. (37)).

Now if losses and noise are to be included, we can assume reservoirs in interaction with the radiation modes and eliminating the reservoir variables in the framework of the Heisenberg or Schrödinger approach in the standard way, we obtain [19, 36, 37] for the time-dependent noise functions $B_1 = B_2 = B$ and D_{12}

$$B = \frac{1}{2}(a_1 + a_2) - 1, \tag{33a}$$

$$D_{12} = \frac{\mathrm{i}}{2}\exp(-\mathrm{i}\omega t + \mathrm{i}\phi)(a_2 - a_1), \tag{33b}$$

$$K = a_1 a_2 - a_1 - a_2 + 1 = (a_1 - 1)(a_2 - 1), \tag{33c}$$

$$a_1 = \frac{\kappa_1}{\kappa_3}[1 - \exp(-\kappa_3 t)] + \exp(-\kappa_3 t), \tag{33d}$$

$$a_2 = \frac{\kappa_2}{\kappa_4}[1 - \exp(\kappa_4 t)] + \exp(\kappa_4 t), \tag{33e}$$

$$\kappa_{1,2} = g \pm \gamma(\langle n_{\mathrm{d}}\rangle + 1), \tag{33f}$$

$$\kappa_{3,4} = 2g \pm \gamma, \tag{33g}$$

γ and $\langle n_{\mathrm{d}}\rangle$ being damping constant and mean number of reservoir oscillators in one mode. We obtain for K

$$K = \frac{g^2 - \gamma^2\langle n_d\rangle^2}{4g^2 - \gamma^2}[1 - \exp(-2gt - \gamma t)] \times [1 - \exp(2gt - \gamma t)] < 0 \tag{34}$$

for all times provided that $g > \gamma/2$, $g > \gamma\langle n_d\rangle$, i.e. above the threshold of the process. We see that K can be negative also below the threshold $g < \gamma/2$ if $0 \leq \langle n_d\rangle < 1/2$. This also holds at the threshold $\gamma = 2g$. On the other hand, the noise functions B_1, B_2, D_{12} can be obtained directly from experimental data [33, 38]–[40].

Using this result we can calculate quasidistributions, generating functions, photon-number distributions and integrated-intensity wave distributions [38]–[40] exhibiting nonclassical behavior, which is equivalent to quantum entanglement [40]. This is clear when applying the Simon condition for separability [41] giving $K(K + 2B + 1) \geq 0$, which is violated in this case (from the positivity of photon-number distributions it always holds the quantum Schwarz inequality $K + B \geq 0$). In fact $K + 2B + 1$ (in general $K + B_1 + B_2 + 1$) is the determinant for antinormal ordering, $K_{\mathcal{A}} = (B_1 + 1)(B_2 + 1) - |D_{12}|^2$, which is always positive. The above formulation can easily be extended to M degrees of freedom (temporal, spatial and polarization) [33, 38, 39] in the spirit of Mandel-Rice formula.

We can also consider stimulated process with initial Gaussian field in the signal and idler modes with the mean number of noise photons $\langle n_{\mathrm{ch1}}\rangle$ and $\langle n_{\mathrm{ch2}}\rangle$ [42]. When performing the averaging with Gaussian distributions, we obtain the characteristic function (32) with the new coefficients

$$B_j^{'} = B_j z + \langle n_{\mathrm{ch}j}\rangle, \quad j = 1,\ 2, \tag{35a}$$

$$D_{12}^{'} = D_{12} z, \tag{35b}$$

$$K^{'} = Kz^2 + z(B_1\langle n_{\mathrm{ch2}}\rangle + B_2\langle n_{\mathrm{ch1}}\rangle) + \langle n_{\mathrm{ch1}}\rangle\langle n_{\mathrm{ch2}}\rangle. \tag{35c}$$

where $z = 1 + \langle n_{\mathrm{ch1}}\rangle + \langle n_{\mathrm{ch2}}\rangle$.

Usually the characteristic function (32) in a single mode of photon pairs is formulated in a matrix form with the help of the complex matrix

$$\hat{A} = \begin{pmatrix} -B_1 & 0 & 0 & D_{12} \\ 0 & -B_1 & D_{12}^* & 0 \\ 0 & D_{12} & -B_2 & 0 \\ D_{12}^* & 0 & 0 & -B_2 \end{pmatrix} \tag{36}$$

and a column vector $\hat{\beta} = (\beta_1, \beta_1^*, \beta_2, \beta_2^*)^T$ (T means transposition) and Hermitian conjugated row vector $\hat{\beta}^\dagger$ as $C_{\mathcal{N}}(\hat{\beta}) = \exp(\hat{\beta}^\dagger\hat{A}\hat{\beta}/2)$. Considerations of separability and entanglement are just based on this matrix [41]. However, we see that the matrix $\hat{A}$ is reducible to the matrix

$$\hat{B} = \begin{pmatrix} -B_1 & D_{12} \\ D_{12}^* & -B_2 \end{pmatrix}, \tag{37}$$

giving the characteristic function in the form $C_{\mathcal{N}}(\hat{\beta}) = \exp(\hat{\beta}^\dagger\hat{B}\hat{\beta})$ with column vector $\hat{\beta} = (\beta_1, \beta_2^*)^T$ just with the determinant $K = B_1B_2 - |D_{12}|^2$ (the degenerate version provides one-mode case with rotating terms specified by a coefficient C instead of D_{12}, then $K = B^2 - |C|^2$ [19]; also a three-mode case [42, 43] can be treated in a matrix form in this way and these two cases are physically the most important). It holds that $\mathrm{Det}[\hat{A}] = K^2$. The determinant K determines the existence or non-existence of the Glauber-Sudarshan quasidistribution [19] and the behavior of the generating function [33, 38] leading to classical or quantum behavior of the integrated-intensity wave distributions. Therefore the nonclassicality and entanglement [41] condition $K < 0$ is the principal one and all other conditions, often algebraically very complicated, are its consequences in this case. The same is valid having a classical behavior and separability given by $K \geq 0$. The generating function is then determined by $\mathrm{Det}[\hat{B}_\lambda]$, where $\hat{B}_\lambda$ is the matrix $\hat{B}$ with $B_{1,2}$ substituted by $B_{1,2} + 1/\lambda_{1,2}$, $\lambda_{1,2}$ being parameters of the generating function [33, 38]. The Simon condition for separability [41] is then written as discussed above as $KK_{\mathcal{A}} \geq 0$. For all this the complex formulation is of the great advantage. This is valid for any s-operator ordering since for the s-entanglement $K_s(K_s + B_{1s} + B_{2s} + 1) < 0$, where $K_s = B_{1s}B_{2s} - |D_{12}|^2$ with $B_{1,2s} = B_{1,2} + (1 - s)/2$. It holds that $K_s > K$, $B_{js} > B_j$ and consequently $K_s + B_{1s} + B_{2s} + 1 > K_{\mathcal{A}} > 0$ and s entanglement occurs for $K_s < 0$, s-separability for $K_s > 0$ and $K_s = 0$ characterizes a diagonal wave distribution related to the s-ordering.

For M equally behaved modes (temporal, spatial and polarization in the spirit of Mandel-Rice formula) we obtain the s-ordered generating function

$$G_s(\lambda_1, \lambda_2) = (1 + \lambda_1 B_{1s} + \lambda_2 B_{2s} + \lambda_1\lambda_2 K_s)^{-M}. \tag{38}$$

Moments of the integrated intensities can be directly derived from moments of the photon numbers:

$$\langle W_i\rangle = \langle n_i\rangle, \tag{39a}$$

$$\langle W_i^2\rangle = \langle n_i^2\rangle - \langle n_i\rangle, \quad i = 1, 2, \tag{39b}$$

$$\langle W_1W_2\rangle = \langle n_1n_2\rangle. \tag{39c}$$

Multi-mode theory of down-conversion developed in [33] using a generalized superposition of signal and noise provides the following relations between the above mentioned experimental quantities and quantum noise coefficients B_1, B_2, D_{12}, and the number of degrees of freedom M:

$$\langle W_j \rangle = MB_j, \quad j = 1,2; \tag{40a}$$

$$\langle (\Delta W_j)^2 \rangle = MB_j^2, \quad j = 1,2; \tag{40b}$$

$$\langle \Delta W_1 \Delta W_2 \rangle = M|D_{12}|^2. \tag{40c}$$

The coefficient B_j gives mean number of photons in mode j, M is the number of modes, and D_{12} characterizes the mutual correlations between signal and idler fields. Inverting relations in Eqs. (40) we arrive at the expressions for parameters B_1, B_2, M, and D_{12}:

$$B_j = \langle (\Delta W_j)^2 \rangle / \langle W_j \rangle, \quad j = 1,2; \tag{41a}$$

$$M_j = \langle W_j \rangle^2 / \langle (\Delta W_j)^2 \rangle, \quad j = 1,2; \tag{41b}$$

$$|D_{12}| = \sqrt{\langle \Delta W_1 \Delta W_2 \rangle / M}. \tag{41c}$$

As follows from Eqs. (41), the number M of modes can be determined from experimental data obtained either from the signal or idler field. This practically means that the experimental data provide two numbers M_1 and M_2 of modes as a consequence of non-perfect alignment of the setup and non-perfect exclusion of noises from the data. On the other hand, only one number M of modes (number of degrees of freedom) appears in the theory [33], because it is assumed that all pairs of mutually entangled signal- and idler-field modes are detected at both detectors. Precise fulfillment of this requirement can hardly be reached under real experimental conditions. However, quantum consistent experimental data really give $M_1 \approx M_2$.

Joint signal-idler photon-number distribution $p(n_1, n_2)$ for multi-chaotic field with M degrees of freedom and composed of photon pairs can be derived in the form [38]:

$$\begin{aligned} p(n_1,n_2) = {} & \frac{1}{\Gamma(M)} \frac{(B_1+K)^{n_1}(B_2+K)^{n_2}}{(1+B_1+B_2+K)^{n_1+n_2+M}} \\ & \times \sum_{r=0}^{\min(n_1,n_2)} \frac{\Gamma(n_1+n_2+M-r)}{r!(n_1-r)!(n_2-r)!} \\ & \times \frac{(-K)^r[(1+B_1+B_2+K)]^r}{[(B_1+K)(B_2+K)]^r}. \end{aligned} \tag{42}$$

The determinant $K = B_1B_2 - |D_{12}|^2$ is crucial for the judgement of classicality or nonclassicality of a field, as discussed above. Negative values of the determinant K mean that a given field cannot be described classically, which is the case of the considered field composed of photon pairs. In Eq. (42), the quantities $B_1 + K$ and $B_2 + K$ cannot be negative and can be considered as characteristics of fictitious noise present in the signal and idler fields, respectively. The theory for an ideal lossless case gives $K = -B_1 = -B_2$ together with the joint photon-number distribution $p(n_1, n_2)$ in the form of diagonal Mandel-Rice distribution. On the other hand inclusion of losses and external noise results in non-diagonal photon-number distribution $p(n_1, n_2)$ as a consequence of the fact that not all detected photons are paired (see Figure 1).

A compound Mandel-Rice formula gives the joint signal-idler photon-number distribution $p(n_1, n_2)$ at the border between the classical and nonclassical characters of the field ($K = 0$):

$$p(n_1,n_2) = \frac{\Gamma(n_1+n_2+M)B_1^{n_1}B_2^{n_2}}{\Gamma(M)n_1!\,n_2!\,(1+B_1+B_2)^{n_1+n_2+M}}. \tag{43}$$

If the number M of modes is large compared to mean values $\langle n_1 \rangle$ and $\langle n_2 \rangle$ (i.e. for B_1, B_2, and $|D_{12}|$ being small) the expression in Eq. (43) can be approximated by product of two Poissonian distributions with no correlations between modes. The joint wave distribution is diagonal in this case.

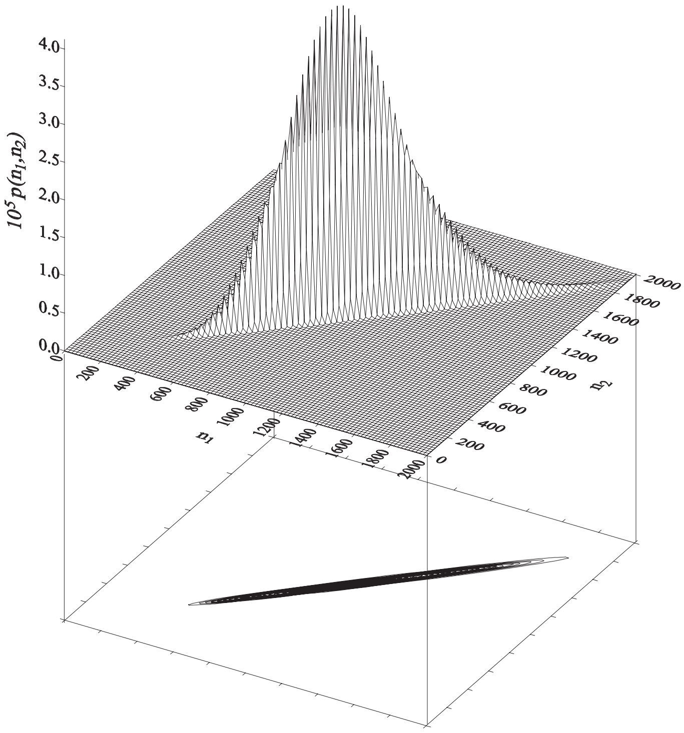


FIG. 1 Joint signal-idler photon-number distribution $p(n_1, n_2)$ (after [39]).

Deviation from an ideal diagonal distribution $p(n_1, n_2)$ caused by losses can be characterized using conditional idler-field photon-number distribution $p_{c,2}(n_2; n_1)$ measured under the condition of detecting n_1 signal photons and determined by the formula:

$$p_{c,2}(n_2; n_1) = p(n_1, n_2) / \sum_{k=0}^{\infty} p(n_1, k). \tag{44}$$

Substitution of Eq. (42) into Eq. (44) leads to the conditional idler-field photon-number distribution $p_{c,2}$ with Fano factor $F_{c,2}$:

$$\begin{aligned} F_{c,2}(n_1) &= 1 + \frac{(1+M/n_1)[(B_2+K)/(1+B_1)]^2 - (K/B_1)^2}{(1+M/n_1)(B_2+K)/(1+B_1) - K/B_1} \\ &\approx 1 + K/B_1. \end{aligned} \tag{45}$$

Approximate expression for the Fano factor $F_{c,2}$ in Eq. (45) (valid for $K \approx -B_2$) indicates that negative values of the determinant K are necessary to observe sub-Poissonian conditional photon-number distributions. Sub-Poissonian distribution emerges from Eq. (42), that is a sum of positive terms in this case. For the ideal lossless case, $K = -B_1 = -B_2$, the Fano factor $F_{c,2}$ equals 0. On the other hand, positive values of the determinant K mean that the sum in Eq. (42) contains large terms with alternating signs (this may lead to numerical errors in summation) and so the conditional distribution $p_{c,2}$ is super-Poissonian. For instance, for K small compared to B_1 we have $F_{c,2} \approx 1 + (B_2 + K)/(1 + B_1)$.

Pairing of photons in the detected signal and idler fields leads to narrowing of the distribution p_- of the difference $n_1 - n_2$ of signal- and idler-field photon numbers:

$$p_-(n) = \sum_{n_1,n_2=0}^{\infty} \delta_{n,n_1-n_2} p(n_1, n_2), \tag{46}$$

where δ denotes Kronecker symbol. If the variance of the difference $n_1 - n_2$ of signal- and idler-field photon numbers is less than the sum of mean photon numbers in the signal and idler fields we speak of sub-shot-noise correlations and characterize them by a coefficient R [43]:

$$R = \frac{\langle [\Delta(n_1 - n_2)]^2 \rangle}{\langle n_1 \rangle + \langle n_2 \rangle} < 1. \tag{47}$$

Joint signal-idler photon-number distribution $p(n_1, n_2)$ and joint signal-idler quasi-distribution $P_{\mathcal{N}}(W_1, W_2)$ of integrated intensities belonging to normally-ordered operators are connected through Mandel's detection equation [19]

$$p(n_1, n_2) = \frac{1}{n_1!\, n_2!} \int_0^\infty dW_1 \int_0^\infty dW_2\, W_1^{n_1} W_2^{n_2} \times \exp(-W_1 - W_2) P_{\mathcal{N}}(W_1, W_2). \tag{48}$$

This relation can be generalized to arbitrary ordering of operators [19, 38] (see Eq. (4)), which can be inverted in terms of series of Laguerre polynomials similarly as in Section 3 (converging to the distribution described by Eq. (42) just for $s \le s_{\rm th}$ given in Eq. (51), otherwise the relation defines a quasidistribution).

Provided that the s-ordered determinant K_s is positive the s-ordered joint signal-idler quasi-distribution $P_s(W_1, W_2)$ of integrated intensities exists as an ordinary function [38] which cannot take on negative values:

$$P_s(W_1, W_2) = \frac{1}{\Gamma(M) K_s^M} \left(\frac{K_s^2 W_1 W_2}{|D_{12}|^2} \right)^{(M-1)/2} \times \exp\left[-\frac{(B_{2s} W_1 / B_{1s} + W_2) B_{1s}}{K_s} \right] \times I_{M-1}\left(2\sqrt{\frac{|D_{12}|^2 W_1 W_2}{K_s^2}} \right). \tag{49}$$

If the s-ordered determinant K_s is negative, the joint signal-idler quasi-distribution P_s of integrated intensities exists in general as a generalized function that can take on negative values or can even have singularities. It can be approximated by the following formula [38]:

$$P_s(W_1, W_2) \approx \frac{A (W_1 W_2)^{(M-1)/2}}{\pi \Gamma(M) (B_{1s} B_{2s})^{M/2}} \exp\left(-\frac{W_1}{2B_{1s}} - \frac{W_2}{2B_{2s}} \right) \times \operatorname{sinc}\left[A \left(\frac{B_{2s}}{B_{1s}} W_1 - W_2 \right) \right]; \tag{50}$$

$\operatorname{sinc}(x) = \sin(x)/x$, $A = (-K_s B_{2s}/B_{1s})^{-1/2}$. Oscillating behavior is typical for the quasi-distribution P_s written in Eq. (50).

There exists a threshold value $s_{\rm th}$ of the ordering parameter s for given values of parameters B_1, B_2, and D_{12} determined by $K_s = 0$ (the joint s-ordered wave distribution is diagonal):

$$s_{\rm th} = 1 + B_1 + B_2 - \sqrt{(B_1 + B_2)^2 - 4K}; \tag{51}$$

$-1 \le s_{\rm th} \le 1$. Quasi-distributions P_s for $s \le s_{\rm th}$ are ordinary functions with non-negative values whereas those for $s > s_{\rm th}$ are generalized functions with negative values and oscillations.

Similarly as for photon numbers we can define a quasi-distribution $P_{s,-}$ of the difference $W_1 - W_2$ of signal- and idler-field integrated intensities as a quantity useful for the description of photon pairing:

$$P_{s,-}(W) = \int_0^\infty dW_1 \int_0^\infty dW_2 \times \delta(W - W_1 + W_2) P_s(W_1, W_2). \tag{52}$$

Quasi-distribution $P_{s,-}$ oscillates and takes on negative values as a consequence of pairwise character of the detected fields if $s \ge s_{\rm th}$.

There exists a relation between variances of the difference of n of signal- and idler-field photon numbers and of the difference of W of signal- and idler-field integrated intensities:

$$\langle [\Delta(n_1 - n_2)]^2 \rangle = \langle n_1 \rangle + \langle n_2 \rangle + \langle [\Delta(W_1 - W_2)]^2 \rangle. \tag{53}$$

According to Eq. (53) negative values of the quasi-distribution $P_{s,-}$ (as well as these of quasi-distribution P_s) are necessary to observe sub-shot-noise correlations in signal- and idler-field photon numbers as described by the condition $R < 1$.

The above results are given for spontaneous process. Similar results, however much more complex, can be obtained for process stimulated by coherent laser fields [33, 44, 45].

We can now illustrate the above results. The joint signal-idler photon-number distribution $p(n_1, n_2)$ determined simply by Eq. (42) for values of experimental parameters given in [39] is shown in Figure 1. Strong correlations in signal-field n_1 and idler-field n_2 photon numbers are clearly visible. Nonzero elements of the joint photon-number distribution $p(n_1, n_2)$ are localized around a line given by the condition $n_1 \approx n_2$ as documented in contour plot in Figure 1.

Joint signal-idler wave quasi-distributions $P_s(W_1, W_2)$ of integrated intensities differ qualitatively according to the value of ordering parameter s ($s_{\rm th} = 0.15$ for the experimental data) (after Eqs. (49) and (50)). Nonclassical character of the detected fields is smooth out ($K_s = 2.66 > 0$) for the value of s equal to 0.1 as shown in Figure 2(a). On the other hand, the value of s equal to 0.2 is sufficient to observe quantum features ($K_s = -2.53 < 0$) in the joint signal-idler quasi-distribution $P_s(W_1, W_2)$ that is plotted in Figure 2(b). In this case oscillations and negative values occur in the graph of the joint quasi-distribution $P_s(W_1, W_2)$.

Similar results can be reported for three-mode parametric processes [40].

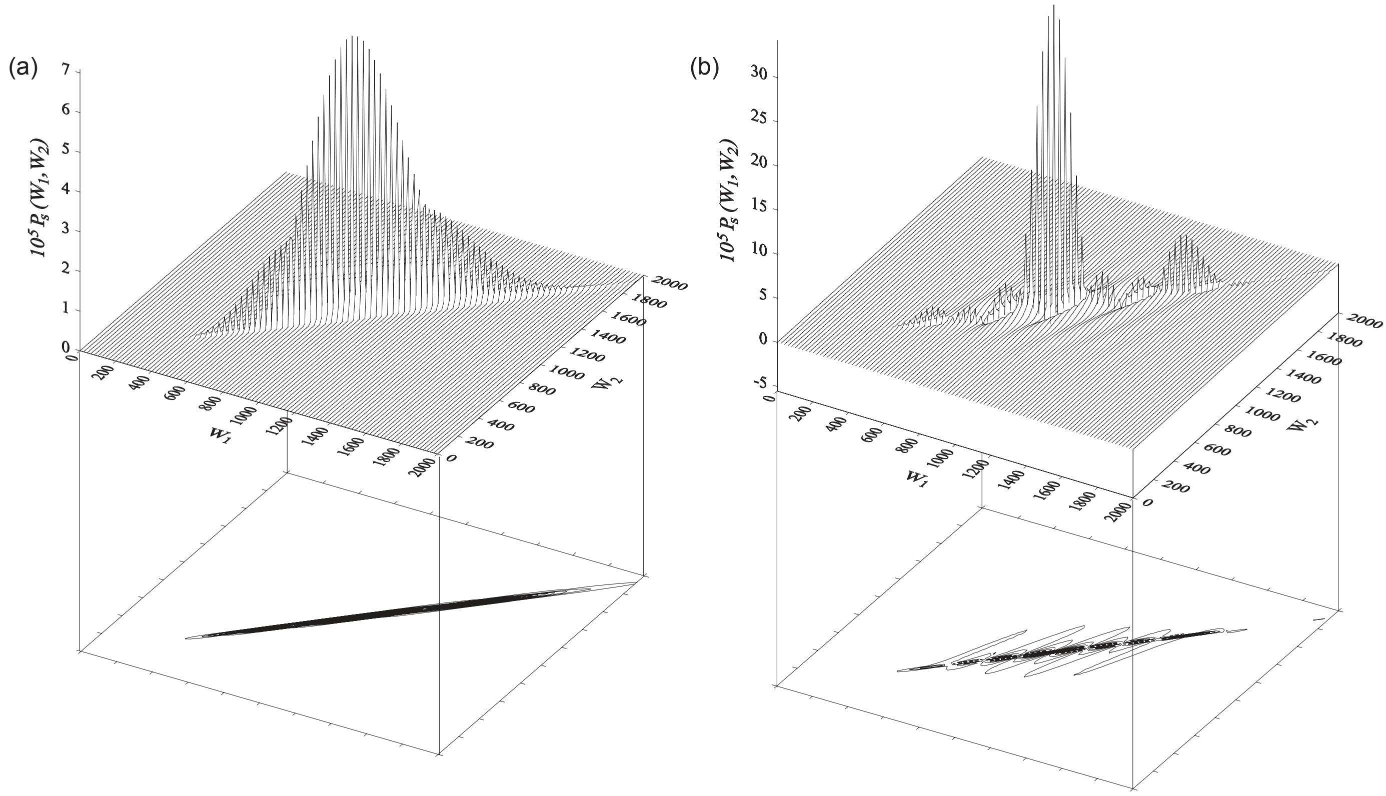


FIG. 2 Joint signal-idler quasi-distributions $P_s(W_1, W_2)$ of integrated intensities of signal (W_1) and idler (W_2) fields for $s = 0.1$ (left) and $s = 0.2$ (right) (after [39]).

8 NONCLASSICALITY AND ENTANGLEMENT IN PARAMETRIC DOWN-CONVERSION

In order to characterize nonclassical behavior (entanglement) of signal and idler beams, we can use the criteria [46]

$$(i) \quad K' = B_1' B_2' - |D_{12}'|^2 < 0, \tag{54a}$$

$$(ii) \quad R = \frac{\langle [\Delta(n_1 - n_2)]^2 \rangle}{\langle n_1 \rangle + \langle n_2 \rangle} = 1 + \frac{\langle [\Delta(W_1 - W_2)]^2 \rangle}{\langle W_1 \rangle + \langle W_2 \rangle} < 1, \tag{54b}$$

$$(iii) \quad S = \frac{\langle (n_1 - n_2)^2 \rangle}{\langle n_1 \rangle + \langle n_2 \rangle} = 1 + \frac{\langle (W_1 - W_2)^2 \rangle}{\langle W_1 \rangle + \langle W_2 \rangle} < 1. \tag{54c}$$

Here $\langle n_j \rangle = \langle W_j \rangle = B_j'$, $\langle (\Delta W_j)^2 \rangle = B_j'^2$, $\langle \Delta W_1 \Delta W_2 \rangle = |D_{12}'|^2$. For $B_1 = B_2 = B$ and $\langle n_{\mathrm{ch1}} \rangle = \bar{n}_1 = \langle n_{\mathrm{ch2}} \rangle = \bar{n}_2 = \bar{n}$ all these criteria are equal. Of course the primary is the first criterion, the second and third ones are derived.

In this way we obtain successively

$$(i) \quad Kz^2 + B_1 z \bar{n}_2 + B_2 z \bar{n}_1 + \bar{n}_1 \bar{n}_2 < 0, \tag{55a}$$

$$(ii) \quad 2Kz^2 + 2B_1 z \bar{n}_1 + 2B_2 z \bar{n}_2 + \bar{n}_1^2 + \bar{n}_2^2 + (B_1 - B_2)^2 z^2 < 0, \tag{55b}$$

$$(iii) \quad Kz^2 + 2B_1 z \bar{n}_1 + 2B_2 z \bar{n}_2 - B_1 z \bar{n}_2 - B_2 z \bar{n}_1 + \bar{n}_1 \bar{n}_2 + (B_1 - B_2)^2 z^2 + (\bar{n}_1 - \bar{n}_2)^2 < 0, \tag{55c}$$

and for $B_1 = B_2 = B$ we obtain

$$(i) \quad (K + B)(\bar{n}_1 + \bar{n}_2) + \bar{n}_1 \bar{n}_2 / (1 + \bar{n}_1 + \bar{n}_2) < -K, \tag{56a}$$

$$(ii) \quad 2(K + B)(\bar{n}_1 + \bar{n}_2) + (\bar{n}_1^2 + \bar{n}_2^2)/(1 + \bar{n}_1 + \bar{n}_2) < -2K, \tag{56b}$$

$$(iii) \quad (K + B)(\bar{n}_1 + \bar{n}_2) + [\bar{n}_1 \bar{n}_2 + (\bar{n}_1 - \bar{n}_2)^2]/(1 + \bar{n}_1 + \bar{n}_2) < -K. \tag{56c}$$

In the pure process $K = -B$ we obtain the earlier results [43]

$$(i) \quad -B(1 + \bar{n}_1 + \bar{n}_2) + \bar{n}_1 \bar{n}_2 < 0, \tag{57a}$$

$$(ii) \quad -2B(1 + \bar{n}_1 + \bar{n}_2) + \bar{n}_1^2 + \bar{n}_2^2 < 0, \tag{57b}$$

$$(iii) \quad -B(1 + \bar{n}_1 + \bar{n}_2) + \bar{n}_1 \bar{n}_2 + (\bar{n}_1 - \bar{n}_2)^2 < 0. \tag{57c}$$

However, these conditions involving the ideal case $K = -B$ are particular compared to more general ones given above for $K + B > 0$ involving losses and noise. For $\bar{n}_1 = \bar{n}_2 = \bar{n}$ all the conditions of nonclassicality are the same

$$B > \frac{\bar{n}^2}{1 + 2\bar{n}}. \tag{58}$$

If one stimulating field is zero, $\bar{n}_1 = \bar{n}$, $\bar{n}_2 = 0$, then [43]

$$(i) \quad B(1 + \bar{n}) > 0, \tag{59a}$$

$$(ii) \quad B > \frac{\bar{n}^2}{2(1 + \bar{n})}, \tag{59b}$$

$$(iii) \quad B > \frac{\bar{n}^2}{(1 + \bar{n})}. \tag{59c}$$

The first condition gives no low bound for B and arbitrary $\bar{n}$. The relative robustness of nonclassicality with respect to the chaotic seed is in agreement with the above result for $g > \gamma/2, \gamma\langle n_d\rangle$. In general, nonclassicality can be supported ($K' < K$) as well as reduced ($K' > K$) by injected chaotic field in dependence on its intensity.

We can illustrate these conditions using Eqs. (33) and (34) including losses and noise. Effects of Langevin forces even for reservoir vacuum ($\langle n_d\rangle = 0$) cause more complicated behavior than it is obtained by simple phenomenological description of losses with the help of the factor $\exp(-\gamma t)$, because $K + B > 0$ even in pure lossy case without noise ($\langle n_d\rangle = 0$) and $K = -B$ holds only for lossless and noiseless case ($\gamma = \langle n_d\rangle = 0$). Both the descriptions can give the same results only if $\gamma t \ll 1$.

From Eqs. (56) we obtain for $\bar{n}_1 = \bar{n}_2 = \bar{n}$

$$\bar{n}(K + B) + \bar{n}^2/2(1 + 2\bar{n}) < -K/2 \qquad (60)$$

for all three conditions, and for $\bar{n}_1 = \bar{n}$, $\bar{n}_2 = 0$ we have successively

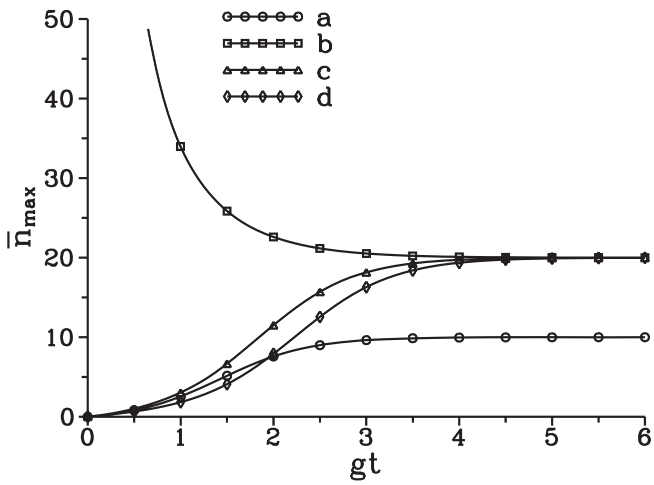
$$(i) \quad \bar{n}(K + B) < -K, \qquad (61a)$$

$$(ii) \quad \bar{n}(K + B) + \bar{n}^2/2(1 + \bar{n}) < -K, \qquad (61b)$$

$$(iii) \quad \bar{n}(K + B) + \bar{n}^2/(1 + \bar{n}) < -K. \qquad (61c)$$

Compared to the spontaneous process we have, in general, bounds of gt for a given $\bar{n}$.

As shown above the spontaneous process produces nonclassical and entangled light for all gt provided that $g > \gamma/2, \gamma\langle n_d\rangle$. For the pure process ($K = -B$) stimulated by means of chaotic light there is generally low bound for B (gt) to generate such light for any value of stimulating intensity [43]. If we consider the process stimulated with chaotic light and including losses and noise ($K + B > 0$), we see from Figure 3 that we have again low bound of B (gt) for nonclassical generation. However, $\bar{n}_{max}$ is saturated and it is decreasing with increasing reservoir noise ($\langle n_d\rangle$) and losses (γ), excepting condition (61)(i) without noise (Fig 3 (top), curve b), where $\bar{n}$ can be arbitrary, however, it gives the upper bound for gt if the values of $\bar{n}$ are higher than the saturated value for curves b, c, d; there is no bound and $\bar{n}$ is arbitrary in the case without losses and noise. For higher $\bar{n}$ we have classical behavior. Asymptotic values of $\bar{n}$ are $2(g/\gamma - \langle n_d\rangle)/(1 + 2\langle n_d\rangle)$ for conditions (61)(i)–(iii) and one half of this value for the condition (60) above threshold (Figure 3). This is valid including $\langle n_d\rangle = 0$ (the effect of lossy vacuum, different from phenomenological introduction of losses and not conserving the pure condition $K = -B$). In Figure 3 we see the maximum values of $\bar{n}$ determining the lowest value of gt from which the field is nonclassical and entangled with noise and losses included (bottom), excepting the curve b without noise (top), as discussed above. This is shown successively for conditions (60) and (61). We see that the determining conditions (60) and (61)(i) are strongest, the other ones are weaker. In Figure 3 $\langle n_d\rangle$ and γ/g are fixed.



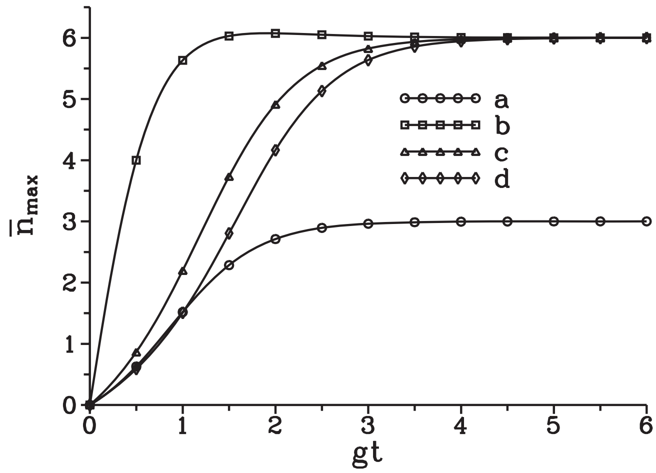


FIG. 3 Conditions (60)(a) and (61)(i)–(iii) (b–d) for nonclassical behavior and entanglement for $\gamma/g = 0.1$, $\langle n_d\rangle = 0$ (top) and $\langle n_d\rangle = 1$ (bottom) (after [46]).

9 CONCLUSIONS

In this review we remember the reasons and beginning for creation of quantum theory coherence and give some results in quantum and nonlinear optics using such quantum description. In particular we discuss quantum operator ordering in relation to reconstruction of quantum states, generalization of classical superposition of signal and noise to include quantum noise, quantum Zeno effect, applications of quantum optical methods to nonlinear couplers and to optical parametric processes from the point of view of joint number and wave distributions and entanglement. A particular attention is paid to nonclassical properties of optical beams. Some results are illustrated on the basis of experimental data.

Finally we mention that the Glauber-Sudarshan diagonal representation stimulated a number of mathematical investigations concerning its existence as a generalized function, as reviewed in the book [47]. Further the quantum optical techniques were applied to various systems, such as light scattering systems [48], periodical systems for effective generation of nonclassical light [49]–[52], etc.

10 ACKNOWLEDGEMENT

This work was supported by Research Project MSM6198959213 and Projects COST 09026 and 1M06002.

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