Effects of partial coherence on frequency combs

Víctor Torres–Company
vtorres@fca.uji.es
Departament de Física, Universitat Jaume I, ES–12080 Castelló, Spain.

Hanna Lajunen
School of Information and Communication Technology, Royal Institute of Technology (KTH), Electrum 229, SE–16440 Kista, Sweden,
Permanent address: Department of Physics and Mathematics, University of Joensuu, P.O. Box 111, FI–80101 Joensuu, Finland.

Ari T. Friberg
School of Information and Communication Technology, Royal Institute of Technology (KTH), Electrum 229, SE–16440 Kista, Sweden.

We give an analytical model to study the deviations from an ideal frequency comb when the effects of random variations, from pulse to pulse, in the amplitude, phase, shape, chirp, and timing jitter are simultaneously taken into account. This model is consistent with the previous experimental research and extends the theoretical work on spectral line widths of mode–locked lasers.[DOI: 10.2971/jeos.2007.07007]

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1 Introduction

The optical energy spectrum of a train of ultrashort pulses consists of multiple longitudinal modes spaced by a frequency equal to the pulse repetition rate. The energy carried by each line is related to the Fourier transform of the pulse envelope. Due to the difference between the group and phase velocities in the laser cavity, there is a constant phase slip from pulse–to–pulse which determines the comb offset. When this offset and repetition rate are stabilized to specific values, the frequency comb can be employed as a ruler for high resolution optical metrology [1, 2].

However, as is well known, various sources of noise can deteriorate the ideal frequency–comb energy spectrum [3]– [7]. In order to assess the stochastic noise phenomena, previous efforts have mainly focused on the mathematical description of the effects of pulse–to–pulse fluctuations in timing jitter and amplitude [8]– [12]. Only very recently, the noise in relative phase has been taken into account [6, 7]. However, there can also be random fluctuations in the intensity profiles, chirping, and temporal durations of the pulses, which have been ignored. It means that the pulse train is an inherently temporally partially coherent wave field. A realization of the random field contains a large number of individual pulses.

In this work, we give a simple analytical model to include simultaneously all these random effects and the timing jitter. Under the assumption of statistical de–correlation, we give a closed–form expression for the energy spectrum which deals with partial coherence and timing jitter separately. We find that the effects of partial coherence contribute to broaden the comb lines, whereas the pulse jitter manifests as a two–sided pedestal of the comb. Our approach is especially useful for any frequency comb produced by actively mode–locked pulses or, alternatively, for combs obtained by external modulation of a continuous wave (CW) laser source [13]. The presented model accounts successfully for the previous experimental researches on the spectral line width of longitudinal modes in mode–locked lasers [5].

2 THEORY

The complex field of the pulse train, $U(t)$, when all noise phenomena are ignored, is [1]

$$U(t) = \sum_{n=-N}^{N} \psi(t-nT) \exp[-i(\omega_c t - n\omega_c T + n\Delta\Phi_{CE})], \quad (1)$$

where ω_c is the carrier frequency, T denotes the fundamental period, $\psi(t)$ is the complex field of each of the $2N+1$ pulses constituting the train, and $\Delta\Phi_{CE}$ denotes the pulse–to–pulse phase shift. To account for the effects of partial temporal coherence and timing jitter, we rewrite Eq. (1) as

$$U(t) = N(t) \sum_{n=-N}^{N} \psi(t-T_n) \exp[-i(\omega_c t - n\omega_c T + n\Delta\Phi_{CE})]. \quad (2)$$

Here $N(t)$ is a dimensionless, statistically stationary, complex random process [14]. Its mutual coherence function is $\Gamma_N(t_1, t_2) = \langle N^*(t_1)N(t_2)\rangle = \Gamma_N(\tau)$, where the angle brackets denote ensemble averaging and $\tau = t_2 - t_1$. The mean intensity of the noise is constant, $\langle |N(t)|^2 \rangle = \Gamma_N(0)$, and without any loss of generality we may assume that this function is normalized, i.e., $\Gamma_N(0) = 1$. The variations in the pulse repetition rate are treated mathematically in Eq. (2) by writing $T_n = nT + \epsilon_n$, where ϵ_n is a small temporal fluctuation that

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accounts for the timing jitter. The specific statistics of ϵ_n depend on the way the laser is mode locked [11, 12]. To keep the approach general we will not restrict ourselves to any particular type of mode–locking process.

We emphasize that since $N(t)$ is a complex random function, it can simultaneously model the temporal fluctuations of the phase, amplitude, energy, and pulse duration, including also the usually ignored chirping and shape variations. In other words, it deals with general temporal partial coherence. Our approach can be considered as a generalization of the classic work of von der Linde [9] to the complex field. Thus we are examining optical energy spectra instead of intensity power spectra. Note that the relative random amplitude fluctuations studied in previous works [11, 12] can be recovered just by restricting $N(t) = a_n$ for $(n-1)T < t < nT$, where a_n is an appropriate real and positive random variable. The phase fluctuation between pulses [6, 7] could be represented similarly by making $N(t)$ a stepwise unimodular complex random process. It is also worth mentioning that, in a different context, the multiplicative noise model has been successfully employed to describe temporal fluctuations taking place in CW lasers that are externally amplitude modulated [15]. This has proven to be useful for determining the maximum bit rate achievable in fiber–optical communication systems that operate by wavelength division multiplexing [16].

The energy spectrum, $S(\omega)$, of the mode–locked laser is defined as [17]

$$S(\omega) = \langle |\tilde{U}(\omega)|^2 \rangle, \tag{3}$$

where $\tilde{U}(\omega)$ is the Fourier transform of Eq. (2). Providing the noise and the timing jitter are uncorrelated the spectrum can be expressed, after some straightforward calculations, in the simple form

$$S(\omega) = W_N(\omega - \omega_c) \otimes F(\omega - \omega_c), \tag{4}$$

where $\otimes$ stands for the convolution operation and

$$F(\omega) = |\tilde{\psi}(\omega)|^2 \sum_{n=-N}^{N} \sum_{m=-N}^{N} \exp[-i(n-m) \times (\omega T + \omega_c T - \Delta\Phi_{CE})] \langle \exp[-i\omega(\epsilon_n - \epsilon_m)] \rangle, \tag{5}$$

where $\tilde{\psi}(\omega)$ is the Fourier transform of $\psi(t)$. In addition, we have used the fact that different frequency components of a stationary process are uncorrelated, i.e., $\langle \tilde{N}^*(\omega)\tilde{N}(\omega')\rangle = W_N(\omega)\delta(\omega' - \omega)$, where δ is the Dirac delta function. Moreover, $W_N(\omega)$ is the spectral density of $N(t)$, which is related to $\Gamma_N(\tau)$ via the Wiener–Khintchine theorem [14]

$$W_N(\omega) = \int \Gamma_N(\tau) \exp(i\omega\tau) d\tau. \tag{6}$$

Eq. (4) is the main result of the present work. It connects separately, through the simple operation of convolution, the effects of partial coherence, given by $W_N(\omega)$, and the effects of pulse jitter, described by $F(\omega)$. Moreover, no assumptions on the mode–locking process are made, which allows us to directly utilize previous models developed for the jitter [11, 12].

To gain further physical insight, we first ignore the timing jitter by setting $\epsilon_n = 0$ for all n. When $N \to \infty$ we find [1] that $F(\omega) = |\tilde{\psi}(\omega)|^2 \sum_{n=-\infty}^{\infty} \delta(\omega + \omega_c - \Delta\Phi_{CE}/T - 2\pi n/T)$ and, in consequence, Eq. (4) becomes

$$S(\omega) = \sum_{n=-\infty}^{\infty} W_N(\omega - \Delta\Phi_{CE}/T - 2\pi n/T) \times |\tilde{\psi}(\Delta\Phi_{CE}/T + 2\pi n/T - \omega_c)|^2. \tag{7}$$

This result shows that the spectral content of a single ideal pulse, $|\tilde{\psi}(\omega)|^2$, is sampled, instead of a comb, by a different sampling function with some spectral width [6, 7]. We note that not only the phase fluctuations but the general partial temporal coherence properties contribute. In this way, every spectral line acquires a finite width given by the spectral density function, $W_N(\omega)$, of the noise $N(t)$. Thus, the line broadening contains the information of the coherence of the pulse train. The deterministic carrier–envelope phase shift only affects by changing the positions of the lines in the same way as in the coherent case. Moreover, when timing jitter is not relevant, the line width is independent of the particular longitudinal mode under consideration, consistently with some previous experimental observations [5].

To study the interplay between timing fluctuations and partial coherence, we first have to pick a specific statistical model for the jitter. We consider a Gaussian statistical distribution with zero mean; it permits us to write Eq. (5) in the following form

$$F(\omega) = |\tilde{\psi}(\omega)|^2 \exp(-\omega^2\epsilon_0^2) \sum_{n=-N}^{N} \sum_{m=-N}^{N} \exp[-i(n-m) \times (\omega T + \omega_c T - \Delta\Phi_{CE})] \exp(\omega^2 \langle \epsilon_m \epsilon_n \rangle), \tag{8}$$

where $\epsilon_0^2 = \langle \epsilon_n^2 \rangle$ for all n. The specific statistics of $\langle \epsilon_n \epsilon_m \rangle$ are determined by the mode–locking process.

First, it is possible that the timing fluctuations are uncorrelated from pulse to pulse, so that $\langle \epsilon_n \epsilon_m \rangle = \epsilon_0^2 \delta_{nm}$, where δ_{nm} is the Kronecker delta function. Such statistical timing jitter can be produced in gain–switched laser diodes [10]. In contrast, for actively mode–locked lasers the timing fluctuations are induced by the intra–cavity modulator. In this case stationary statistics are widely assumed [11], i.e., $\langle \epsilon_n \epsilon_m \rangle = \langle \epsilon_0 \epsilon_{|n-m|} \rangle = G(|n-m|)$. In this way, the correlation between the pulse–period fluctuations only depends on the absolute time interval between the considered pulses. Finally, for passively mode–locked lasers the stationarity assumption no longer holds [12]. Of course, such a jitter can still be described by our model through Eqs. (4) and (8). With no loss of generality, we demonstrate the model with the uncorrelated and stationary cases.

3 NUMERICAL EXAMPLES

For numerical examples, we consider an infinite pulse train with 40 GHz repetition rate and Gaussian envelope $\psi(t)$. The root–mean–square temporal length of the pulse intensity is taken to be about 1.12 ps. The central frequency is $\omega_c/2\pi \approx 193.55$ THz, corresponding to the wavelength 1.55 μm. We also take $\omega_c T - \Delta\Phi_{CE} = 0$. Different values would shift, as

[1] With use of the identity $\sum_{n=-\infty}^{\infty} \exp(-i\omega T n) = \sum_{n=-\infty}^{\infty} \delta(\omega - 2\pi n/T)$.

in the coherent case, the comb structure by an amount between 0 and $2\pi/T$. Based on previous experimental results [4], the noise spectrum $W_N(\omega)$ is chosen as a Lorentzian function with line width $\Delta\omega$. Note that according to Eq. (6) this corresponds to a mutual coherence function $\Gamma_N(\tau)$ with exponential decay. Thus the coherence time of the pulse train is roughly given by the inverse of the line width. Again, this is closely connected to the single–mode operation of CW lasers.

Figure 1 illustrates the optical power spectra of pulse trains with different states of coherence when there is no timing jitter.

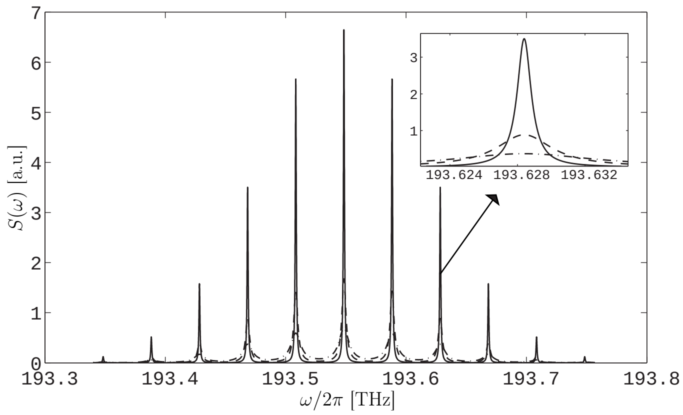

FIG. 1 Energy spectra of mode-locked lasers with different states of coherence and no timing jitter. The widths of the Lorentzian noise spectra are $\Delta\omega = 2\pi f$, where $f = 0.5$ GHz (solid line), 2 GHz (dashed line), and 5 GHz (dash-dotted line). A closer look of one of the comb lines is shown in the inset.

The comb lines are seen to become wider as the amount of noise in the train increases. In Figure 2 the same pulse trains are assumed to have uncorrelated timing jitter with $\epsilon_0 \approx 79$ fs, which corresponds to a 0.3 % deviation from the mean period.

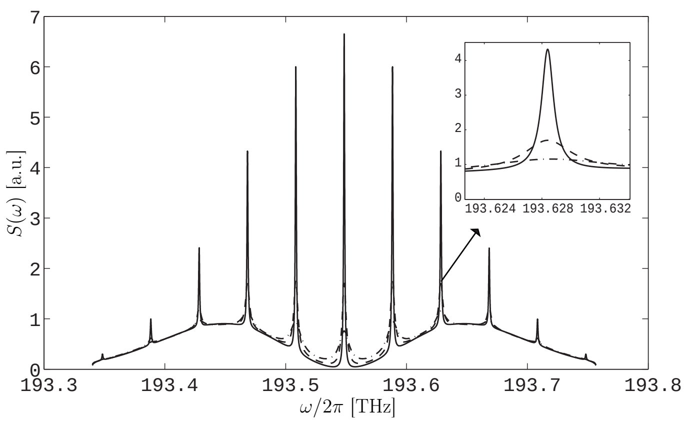

FIG. 2 Energy spectra of mode-locked lasers with different states of coherence and uncorrelated timing jitter. The widths of the Lorentzian noise spectra are as in Figure 1. A close-up of one of the comblines is shown in the inset.

The line–width variations depending on the state of coherence remain similar as those in Figure 1, but the timing jitter changes the shape of the spectrum by creating additional pedestals on both sides of the central frequency. Finally, in Figure 3 we consider pulse trains with the same noise functions but more general stationary timing jitter. The parameter ϵ_0 is as in the uncorrelated case, and the correlation of the jitter between the separate pulses is assumed to be given by function [8,11] $G(n) = \epsilon_0^2 \exp(-\alpha n)$, with $\alpha = 1$.

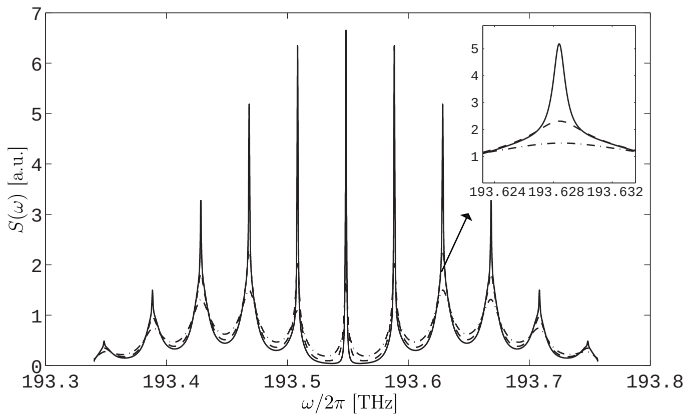

FIG. 3 Energy spectra of mode-locked lasers with different states of coherence and stationary timing jitter. The widths of the Lorentzian noise spectra are as in Figure 1. A close-up of one of the comblines is shown in the inset.

In this case, the jitter causes not only the pedestal parts to the spectrum, but also changes to the widths and shapes of the comb lines. It should also be noted that the envelope function is no longer the same Gaussian as with no timing jitter.

4 DISCUSSION AND CONCLUSIONS

Nevertheless, as predicted by the theory and in accordance with previous approaches [6], the central line of the comb is not altered by the presence of jitter. When there is no overlap between neighboring lines the coherence time, t_c, can be defined as the inverse of the bandwidth $\Delta\omega_c$ of that line, given for instance by (see Eq. (4.3–68) in Ref. [14])

$$\Delta\omega_c^2 = \frac{\int_{\omega_-}^{\omega_+} (\omega-\omega_0)^2 S^2(\omega)\mathrm{d}\omega}{\int_{\omega_-}^{\omega_+} S^2(\omega)\mathrm{d}\omega}, \tag{9}$$

where now $\omega_0 = \omega_c - \Delta\Phi_{CE}/T$ and $\omega_\pm = \omega_0 \pm \pi/T$. It is worth mentioning that the coherence time of frequency combs is often defined as the inverse of the maximum frequency at which the energy of the phase–noise spectral density of the spectral line corresponding to the offset has accumulated a phase of 1 rad [1]. However, although quite practical in terms of experimental implementation, this definition of coherence only takes into account the carrier–envelope phase coherence, i.e., the random changes in phase. It is not an appropriate measure of the general temporal optical coherence. In order to implement a measurement of the quantity described in Eq. (9), an approach such as the one described in Ref. [5] should be adopted. A proper filter on the central line first converts the pulsed laser into CW radiation. Thereafter, a delayed self–heterodyne detection method is used, which allows to extract the optical energy spectrum.

Additionally, we have also been studying the case in which the timing jitter and the temporal partial coherence are not statistically independent. In this case the energy spectrum remains practically the same as that plotted in Figure 3, but spikes appear at the top of the spectral lines, apart from the central line. So, the procedure to measure the coherence time as the inverse of the central–line width given by Eq. (9) remains valid. We find that the height of the spikes is proportional to the correlation between the strength of the jitter and noise. Complete analysis of this more complicated situation is presented elsewhere [18].

In summary, the effects of temporal partial coherence and timing jitter have been included in the description of mode–locked frequency combs. We have shown that both types of noise influence the ideal comb in their own distinct ways. Our mathematical model is analytical and consistent with previous experimental work on the line widths of mode–locked lasers. The presented theoretical approach opens up possibilities to measure the coherence time of pulse trains and, further, to generate coherently stabilized frequency combs, which would increase the spectral resolution.

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