Transmission coefficient of a one-dimensional multi-layer medium from a sum over all light-rays.

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The transmission coefficient of a one-dimensional multi-layer medium is obtained by summing the amplitude coefficients of all individual transmitted light-rays. As compared to the transfer-matrix method, the sum-over-all-light-rays derivation results in a very intelligible expression, which is interesting from a theoretical point of view. Exactly as in the case of a single-layer medium, the sum of all light-rays through the multi-layer medium is obtained from a geometric series. The system does not have to be periodic; the layers have arbitrary physical lengths and each layer has an arbitrary electromagnetic response. [DOI: 10.2971/jeos.2008.08013]

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1 INTRODUCTION

The propagation of electromagnetic fields in layered, or piecewise homogeneous media, is intensively studied in the field of optics [1]- [4]. The relatively recent interest in photonic crystals [2] has renewed and enhanced the research of this subject.

The quantitative description of the transmission and reflection of electromagnetic waves through/against multi-layer systems is given by the transmission and reflection coefficients of these media. These coefficients give the ratio of the electric field amplitudes of the transmitted or reflected wave to the electric field amplitude of the incident wave. Usually, these coefficients are calculated with the transfer-matrix method [1,5], yielding rather complicated and unintelligible expressions, especially when the number of layers exceeds one. As we will show in this paper, a calculation of the transmission coefficient as the sum of the amplitude coefficients of all individual transmitted light-rays results in a much simpler expression. It turns out that for a multi-layer medium, the structure of this expression remains the same as it is for a single-layer system. Key point in the calculation is the introduction of an efficient basis for the paths along which the transmitted light-rays propagate. For the transmitted field, this decomposition is rather straightforward, because all individual transmitted light-rays have one single path in common. For the reflected field, the situation is slightly more difficult and we have not yet been able to derive the reflection coefficient of the multi-layer medium by summing the amplitude coefficients of all reflected light-rays. It is however to be expected that the expression for the reflection coefficient will also emerge similarly intelligible when it is calculated via the sum-over-all-light-rays.

This paper has been organized as follows. First, the medium is modeled in Section 2. Then, in Section 3, the amplitude coefficients that belong to the basic physical processes of the individual light-rays within the medium are given. In Section 4, a basis is formed for the paths along which the transmitted light-rays propagate within the medium. Section 5 gives the possible sequences in which these various basic path elements can be taken by the light-rays. With the use of this basis and combinatorics, the sum of all transmitted light-rays, and therewith the transmission coefficient, is obtained in Section 6. A brief conclusion is given in Section 7.

2 MODEL FOR THE MEDIUM

Our model for the one-dimensional, stratified, piecewise homogeneous medium is the multi- or N-layer medium, with $N = 0, 1, \ldots$, which has been depicted in Figure 1. The response of the system to an electromagnetic field varies stepwise along the x-axis. Including the two homogeneous sub-

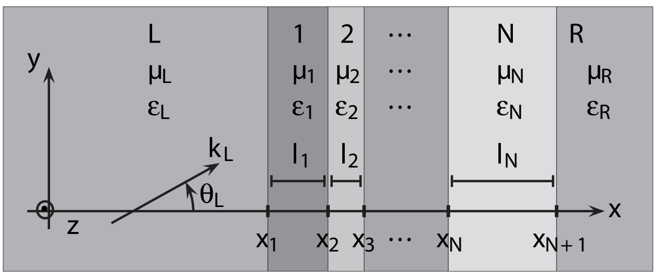

FIG. 1 The one-dimensional, rectangular N-layer medium. In the picture, l_q denotes the physical length of layer q and ϵ_q and μ_q are respectively the permittivities and permeabilities of homogeneous subspace q. Also shown is the wave-vector $\mathbf{k}_L$ of the plane-wave incident field.

spaces that bound the N-layer medium from the left and from the right, there are in total $N + 2$ homogeneous subspaces. As

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has been indicated in Figure 1, these subspaces have been labeled as $q = L, 1, \dots, N, R$ from left to right. The positions of the interfaces $i = 1, \dots, N+1$ that bound the subspaces are at $x = x_i$ and the interfaces 1 and $N+1$ are respectively called the entrance and exit interfaces. The subspaces 1 to N are the actual layers of the N-layer medium and the physical length of layer q is equal to

$$l_q = x_{q+1} - x_q. \tag{1}$$

The response of the multi-layer system and the two surrounding media to the electromagnetic field is taken to be causal, linear and isotropic and in each subspace it is homogeneous. The analysis allows for dispersion and absorption, so the absolute permittivity and absolute permeability in subspace q, which are denoted respectively as ϵ_q and μ_q, can be complex functions of frequency. Now that the medium has been modeled, the effect of this medium on an electromagnetic field will be analyzed in the following section.

3 ELECTROMAGNETIC FIELD IN THE MEDIUM

In this section, it will be shown how the amplitude of the electric field of a light-ray is affected by the elementary physical processes that the light-ray can perform within the multi-layer system. These processes are propagation in the homogeneous layers, and transmission and reflection at the interfaces. The temporal Fourier transforms of the electric and magnetic field vectors are respectively denoted by $\widetilde{\mathbf{E}}_q$ and $\widetilde{\mathbf{H}}_q$. In Eqs. (A.3) and (A.2) of Appendix A the explicit formulae are given for respectively the temporal Fourier transform and its inverse. The amplitude of the Fourier transform of the electric field vector in subspace q is given by

$$\widetilde{E}_q = \sqrt{\widetilde{\mathbf{E}}_q \cdot \widetilde{\mathbf{E}}_q}. \tag{2}$$

Whenever there exists no ambiguity, we will, for brevity, refer to the amplitude of a Fourier transform as the amplitude. The relative amplitudes of the reflected and transmitted electric fields at interface q are given by the Fresnel coefficients, which are defined as

$$r_{q-1,q} \equiv \left(\widetilde{E}^{(l)}_{q-1} / \widetilde{E}^{(r)}_{q-1}\right)\Big|_{\widetilde{E}^{(l)}_q = 0}, \tag{3a}$$

$$r_{q,q-1} \equiv \left(\widetilde{E}^{(r)}_{q} / \widetilde{E}^{(l)}_{q}\right)\Big|_{\widetilde{E}^{(r)}_{q-1} = 0}, \tag{3b}$$

$$t_{q-1,q} \equiv \left(\widetilde{E}^{(r)}_{q} / \widetilde{E}^{(r)}_{q-1}\right)\Big|_{\widetilde{E}^{(l)}_q = 0}, \tag{3c}$$

$$t_{q,q-1} \equiv \left(\widetilde{E}^{(l)}_{q-1} / \widetilde{E}^{(l)}_{q}\right)\Big|_{\widetilde{E}^{(r)}_{q-1} = 0}, \tag{3d}$$

where all the amplitudes are evaluated at $x = x_q$ and at a fixed y-position and where the superscripts l and r refer to the amplitudes of respectively left- and rightwards propagating fields, which will be explicated below. The transformed fields satisfy the Helmholtz equation (see Appendix A),

$$\left(\nabla^2 + \mathbf{k}_q^2\right) \begin{Bmatrix} \widetilde{\mathbf{E}}_q \\ \widetilde{\mathbf{H}}_q \end{Bmatrix} = 0, \tag{4}$$

with $\mathbf{k}_q^2 = \omega^2 \epsilon_q \mu_q$. The plane of incidence is taken as the xy-plane and the applied field is a plane wave, incident from the left on the medium propagating in the rightwards direction under a real angle $\theta = \theta_L$ with the x-axis, see Figure 1. Snell's law [4] implies that in subspace q the wave-vector of the rightwards propagating field that results from the applied field is given by

$$\mathbf{k}_q = \hat{\mathbf{x}}\omega\sqrt{\epsilon_q\mu_q - \epsilon_L\mu_L \sin^2\theta_L} + \hat{\mathbf{y}}\omega\sqrt{\epsilon_L\mu_L}\sin\theta_L, \tag{5}$$

where, for $q = L$, this gives the wave-vector of the applied field. The square roots in Eq. (5) are understood to have a positive real part. In every subspace, we allow for fields that propagate in both directions along the x-axis because of the possibility of reflections against the interfaces and the two electric field solutions of Eq. (4) that represent TE-polarized left- and rightwards propagating fields are respectively given by

$$\widetilde{\mathbf{E}}^{(l)}_q(\omega; x, y) = \hat{\mathbf{z}}\,\widetilde{A}^{(l)}_q(\omega)\exp\left(-\mathrm{i}k_{q,x}\left(x - x_q\right) + \mathrm{i}k_{q,y}y\right), \tag{6a}$$

$$\widetilde{\mathbf{E}}^{(r)}_q(\omega; x, y) = \hat{\mathbf{z}}\,\widetilde{A}^{(r)}_q(\omega)\exp\left(\mathrm{i}k_{q,x}\left(x - x_q\right) + \mathrm{i}k_{q,y}y\right), \tag{6b}$$

where $k_{q,x}$ and $k_{q,y}$ denote respectively the x- and y-components of $\mathbf{k}_q$. According to Eqs. (A.1a) and (A.4b), the components of the magnetic fields that belong to the TE-polarized solutions of Eqs. (6) are given by

$$\begin{aligned}\widetilde{\mathbf{H}}^{(l)}_q(\omega; x, y) &= \hat{\mathbf{x}}\left(k_{q,y}/\mu_q\omega\right)\widetilde{E}^{(l)}_q(\omega; x, y) \\ &\quad + \hat{\mathbf{y}}\left(k_{q,x}/\mu_q\omega\right)\widetilde{E}^{(l)}_q(\omega; x, y),\end{aligned} \tag{7a}$$

$$\begin{aligned}\widetilde{\mathbf{H}}^{(r)}_q(\omega; x, y) &= \hat{\mathbf{x}}\left(k_{q,y}/\mu_q\omega\right)\widetilde{E}^{(r)}_q(\omega; x, y) \\ &\quad + \hat{\mathbf{y}}\left(-k_{q,x}/\mu_q\omega\right)\widetilde{E}^{(r)}_q(\omega; x, y).\end{aligned} \tag{7b}$$

The coefficients $\widetilde{A}^{(l)}_q$ and $\widetilde{A}^{(r)}_q$ in Eqs. (6) and (7) are determined by the Fourier transforms of the tangential electric and magnetic fields at the interfaces as

$$\widetilde{A}^{(l)}_q(\omega) = \frac{1}{2}\left(\widetilde{E}_q(\omega; x_q, 0) + \left(\mu_q\omega/k_{q,x}\right)\widetilde{H}_{q,y}(\omega; x_q, 0)\right), \tag{8a}$$

$$\widetilde{A}^{(r)}_q(\omega) = \frac{1}{2}\left(\widetilde{E}_q(\omega; x_q, 0) - \left(\mu_q\omega/k_{q,x}\right)\widetilde{H}_{q,y}(\omega; x_q, 0)\right), \tag{8b}$$

where $\widetilde{H}_{q,y}$ is the y-component of the total magnetic field. From Eqs. (6) and (7), it follows that continuity of the tangential electric and magnetic fields at interface q gives respectively

$$\pi_{q-1}\widetilde{A}^{(l)}_{q-1} + \pi^{-1}_{q-1}\widetilde{A}^{(r)}_{q-1} = \widetilde{A}^{(l)}_q + \widetilde{A}^{(r)}_q, \tag{9a}$$

$$\begin{aligned}&\left(k_{q-1,x}/\mu_{q-1}\omega\right)\left(\pi_{q-1}\widetilde{A}^{(l)}_{q-1} - \pi^{-1}_{q-1}\widetilde{A}^{(r)}_{q-1}\right) \\ &\qquad = \left(k_{q,x}/\mu_q\omega\right)\left(\widetilde{A}^{(l)}_q - \widetilde{A}^{(r)}_q\right),\end{aligned} \tag{9b}$$

where we have introduced

$$\pi_q \equiv \exp\left(\mathrm{i}k_{q,x}l_q\right). \tag{10}$$

Hence, from Eqs. (9) and Eqs. (3), it follows that for TE-polarization and for $q' = q \pm 1$,

$$r_{qq'} = \frac{\mu_q^{-1}\sqrt{\epsilon_q\mu_q - \epsilon_L\mu_L \sin^2\theta_L} - \mu_{q'}^{-1}\sqrt{\epsilon_{q'}\mu_{q'} - \epsilon_L\mu_L \sin^2\theta_L}}{\mu_q^{-1}\sqrt{\epsilon_q\mu_q - \epsilon_L\mu_L \sin^2\theta_L} + \mu_{q'}^{-1}\sqrt{\epsilon_{q'}\mu_{q'} - \epsilon_L\mu_L \sin^2\theta_L}}, \tag{11a}$$

$$t_{qq'} = \frac{2\mu_q^{-1}\sqrt{\epsilon_q\mu_q - \epsilon_L\mu_L \sin^2\theta_L}}{\mu_q^{-1}\sqrt{\epsilon_q\mu_q - \epsilon_L\mu_L \sin^2\theta_L} + \mu_{q'}^{-1}\sqrt{\epsilon_{q'}\mu_{q'} - \epsilon_L\mu_L \sin^2\theta_L}}, \tag{11b}$$

The admitted TM-or p-polarized magnetic field solutions of Eq. (4) that represent respectively left- and rightwards propagating fields are given by

$$\widetilde{\mathbf{H}}_q^{(l)}(\omega; x, y) = \hat{\mathbf{z}}\, \widetilde{B}_q^{(l)}(\omega) \exp\left(-\mathrm{i}k_{q,x}\left(x - x_q\right) + \mathrm{i}k_{q,y}y\right), \tag{12a}$$

$$\widetilde{\mathbf{H}}_q^{(r)}(\omega; x, y) = \hat{\mathbf{z}}\, \widetilde{B}_q^{(r)}(\omega) \exp\left(\mathrm{i}k_{q,x}\left(x - x_q\right) + \mathrm{i}k_{q,y}y\right), \tag{12b}$$

From Eqs. (A.1b) and (A.4a), it follows that the components of the electric field that belong to the TM-polarized solutions of Eqs. (12) are given by

$$\begin{aligned}\widetilde{\mathbf{E}}_q^{(l)}(\omega; x, y) &= \hat{\mathbf{x}}\left(-k_{q,y}/\epsilon_q\omega\right)\widetilde{H}_q^{(l)}(\omega; x, y)\\ &\quad + \hat{\mathbf{y}}\left(-k_{q,x}/\epsilon_q\omega\right)\widetilde{H}_q^{(l)}(\omega; x, y), \end{aligned}\tag{13a}$$

$$\begin{aligned}\widetilde{\mathbf{E}}_q^{(r)}(\omega; x, y) &= \hat{\mathbf{x}}\left(-k_{q,y}/\epsilon_q\omega\right)\widetilde{H}_q^{(r)}(\omega; x, y)\\ &\quad + \hat{\mathbf{y}}\left(k_{q,x}/\epsilon_q\omega\right)\widetilde{H}_q^{(r)}(\omega; x, y), \end{aligned}\tag{13b}$$

where $\widetilde{H}_q^{(l)}$ and $\widetilde{H}_q^{(r)}$ are the amplitudes of the left- and rightwards propagating magnetic fields, these amplitudes are defined similarly as the amplitude of the electric field in Eq. (2). The coefficients $\widetilde{B}_q^{(l)}$ and $\widetilde{B}_q^{(r)}$ in Eqs. (12) and (13) are determined by the Fourier transforms of the tangential electric and magnetic fields at the interfaces as

$$\widetilde{B}_q^{(l)}(\omega) = \frac{1}{2}\left(\widetilde{H}_q\left(\omega; x_q, 0\right) - \left(\epsilon_q\omega/k_{q,x}\right)\widetilde{E}_{q,y}\left(\omega; x_q, 0\right)\right), \tag{14a}$$

$$\widetilde{B}_q^{(r)}(\omega) = \frac{1}{2}\left(\widetilde{H}_q\left(\omega; x_q, 0\right) + \left(\epsilon_q\omega/k_{q,x}\right)\widetilde{E}_{q,y}\left(\omega; x_q, 0\right)\right). \tag{14b}$$

From Eqs. (13) and (12), it follows that continuity of the tangential electric and magnetic fields at interface q gives respectively

$$\begin{aligned}\left(k_{q-1,x}/\epsilon_{q-1}\omega\right)&\left(\pi_{q-1}\widetilde{B}_{q-1}^{(l)} - \pi_{q-1}^{-1}\widetilde{B}_{q-1}^{(r)}\right)\\ &= \left(k_{q,x}/\epsilon_q\omega\right)\left(\widetilde{B}_q^{(l)} - \widetilde{B}_q^{(r)}\right), \end{aligned}\tag{15a}$$

$$\pi_{q-1}\widetilde{B}_{q-1}^{(l)} + \pi_{q-1}^{-1}\widetilde{B}_{q-1}^{(r)} = \widetilde{B}_q^{(l)} + \widetilde{B}_q^{(r)}, \tag{15b}$$

and with Eqs. (3), it can be found that for TM-polarized fields, the Fresnel coefficients take on the expressions

$$r_{qq'} = \frac{\epsilon_{q'}^{-1}\sqrt{\epsilon_{q'}\mu_{q'} - \epsilon_L\mu_L \sin^2\theta_L} - \epsilon_q^{-1}\sqrt{\epsilon_q\mu_q - \epsilon_L\mu_L \sin^2\theta_L}}{\epsilon_q^{-1}\sqrt{\epsilon_q\mu_q - \epsilon_L\mu_L \sin^2\theta_L} + \epsilon_{q'}^{-1}\sqrt{\epsilon_{q'}\mu_{q'} - \epsilon_L\mu_L \sin^2\theta_L}}, \tag{16a}$$

$$t_{qq'} = \sqrt{\frac{\epsilon_q\mu_{q'}}{\epsilon_{q'}\mu_q}}\; \frac{2\epsilon_q^{-1}\sqrt{\epsilon_q\mu_q - \epsilon_L\mu_L \sin^2\theta_L}}{\epsilon_q^{-1}\sqrt{\epsilon_q\mu_q - \epsilon_L\mu_L \sin^2\theta_L} + \epsilon_{q'}^{-1}\sqrt{\epsilon_{q'}\mu_{q'} - \epsilon_L\mu_L \sin^2\theta_L}}, \tag{16b}$$

According to Eqs. (2), (6) and (13), the electric field amplitudes satisfy, both for TE- and for TM-polarization,

$$\widetilde{E}_q^{(l)}\left(\omega; x_q, y\right) = \pi_q\widetilde{E}_q^{(l)}\left(\omega; x_{q+1}, y\right), \tag{17a}$$

$$\widetilde{E}_q^{(r)}\left(\omega; x_{q+1}, y\right) = \pi_q\widetilde{E}_q^{(r)}\left(\omega; x_q, y\right), \tag{17b}$$

where π_q has been defined in Eq. (10). Hence, propagation of a light-ray from interface $q+1$ to interface q or vice versa results, under evaluation at the same y-positions at both interfaces, in an amplitude coefficient of π_q. Note that this amplitude factor depends on the angle of incidence θ_L.

Summarizing, the amplitude of the electric component of the electromagnetic field is affected as follows by the multi-layer medium. Propagation within subspace q from interface q to interface $q+1$ or vice versa gives the amplitude a factor π_q of Eq. (10), which holds for both TE- and TM-polarization. Reflection or transmission at interface q results in an amplitude factor given by the Fresnel coefficients of Eqs. (11) for TE- and Eqs. (16) for TM-polarized fields. The effects of the various elementary processes of propagation, reflection and transmission on the electric field amplitude of a light-ray in interaction with the multi-layer medium have now been given, and in the following section the main part of our work begins. This is deriving an expression that gives all pathways for the light-rays that contribute to the transmitted field.

4 PATH DECOMPOSITION

As a consequence of the reflections against interfaces there is an infinite number of paths within the medium along which the applied field propagates from the entrance to the exit interface. We have found a basis from which all these paths can be obtained efficiently. The first step in the path decomposition is to observe that every continuous path from the entrance to the exit interface of the system can be cut into two parts. One part of this path is the *direct* path, which is the continuous path straight from the entrance to the exit interface of the medium. Along this path there are no reflections and the amplitude coefficient that belongs to the direct path is equal to

$$t_N^{(0)} = t_{01}\prod_{q=1}^{N} t_{q,q+1}\pi_q, \tag{18}$$

where $t_{qq'}$ are the Fresnel transmission coefficients and π_q the propagation factors from the previous section. The other part in the decomposition of the path of a generic transmitted light-ray consists of *detour* paths, which are simply the deviations from the direct path. The transmission coefficient t_N is the sum of the amplitude coefficients of the transmitted light-rays along all possible paths. Continuity of the paths implies that the path of every possible transmitted light-ray always has, at least effectively, the direct path in common. The amplitude coefficient of the direct path, Eq. (18), must therefore appear in the transmission coefficient as a common factor and we write

$$t_N = t_N^{(0)} \delta_N, \tag{19}$$

where δ_N denotes the factor that is equal to the sum of the amplitude coefficients of the light-rays along all possible detour paths. The decomposition into direct and detour paths has been illustrated in Figure 2a for an arbitrary transmitted light-ray through a multi-layer medium with $N = 4$.

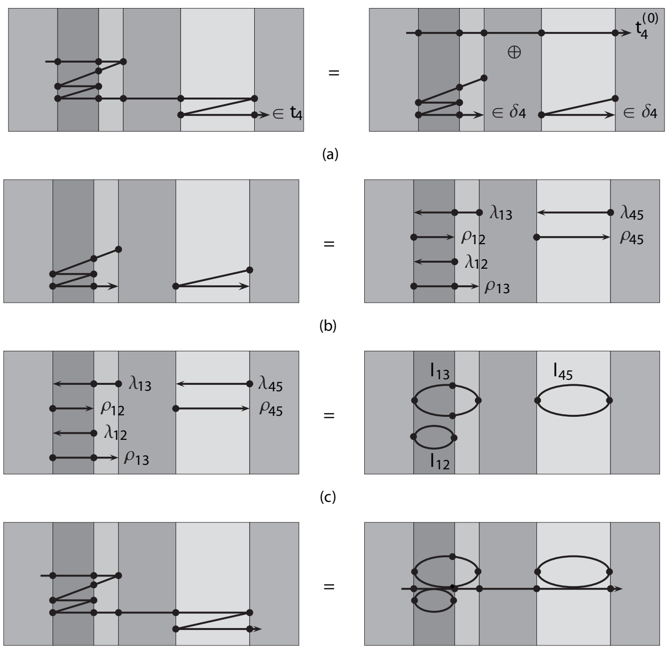

FIG. 2 Illustration of the path decomposition of an arbitrary transmitted light-ray in a multi-layer medium with $N = 4$. The directions of the path-lines do not coincide with the propagation direction of the light-ray. Dots indicate the inclusion of Fresnel coefficients along the path. (a) Decomposition into a direct path and detour paths. (b) Decomposition of detour paths into reflection-induced translations. (c) Decomposition of oppositely directed reflection-induced translations into loops. (d) Net result of Figures 2a-c.

Now we will form a basis for the detour paths. Along each detour path, the light-ray performs a sequence of translations between two interfaces. Each of these translations is initiated by a reflection against an interface and the end of a translation is either just before the following reflection or at the point where the light-ray continues its propagation on the direct path. The translation also includes transmission through intermediate interfaces if the leftmost and the rightmost interfaces along the translation are not neighboring ones. For the translation between a given pair of two different interfaces there are two possibilities. Either it starts with a reflection against the leftmost interface, followed by a rightwards translation to the rightmost interface, or it starts with a reflection against the rightmost interface, followed by leftwards translation to the leftmost interface. The associated amplitude coefficients of these right- and leftwards translations between the interfaces p and $q > p$, are given by respectively

$$\rho_{pq} = r_{p,p-1}\pi_p \prod_{s=p+1}^{n-1} t_{s-1,s}\pi_s, \tag{20a}$$

$$\lambda_{pq} = r_{q-1,q}\pi_{q-1} \prod_{s=p}^{n-2} t_{s+1,s}\pi_s, \tag{20b}$$

where $r_{pp'}$ are the Fresnel reflection coefficients from the previous section. The decomposition of the detour paths into reflection-induced left- and rightwards translations has been illustrated in Figure 2b, for the example path of Figure 2a. Since for a given transmitted light-ray, the propagation along the direct path accounts for the translation through the medium from left to right, the net axial field translation in the sum of all detour paths of this light-ray should be zero. In every detour path, each reflection-induced translation in the leftwards direction between two interfaces should therefore at some stage be, at least effectively, compensated with the reflection-induced translation rightwards. The reflection-induced translations effectively come in oppositely directed pairs, and a basis set for all detour paths can be formed with these pairs. The pair of oppositely directed reflection-induced translations between interfaces p and q give the field an amplitude coefficient

$$l_{pq} = \rho_{pq}\lambda_{pq}. \tag{21}$$

Since these combined translations start and end on the same interface, their path closes in the axial direction and we are actually considering a basis of loops. The set of different loops in the N-layer medium is given by

$$\{l_{pq}\}_N = \sum_{s=2}^{N+1}\sum_{t=1}^{s-1} l_{ts}. \tag{22}$$

Note that this set contains $N(N+1)/2$ elements. The combination of the oppositely directed pairs of reflection-induced translations into loops has been illustrated in Figure 2c, for the example detour paths of Figure 2c. The net result of the steps that have been illustrated in Figures 2a to Figure 2c is given in Figure 2d. With Eq. (22), the basis elements of the detour paths have been identified as loops, or back-and-forth scattering events between interfaces. In the following section, we obtain the number of possible realizations of a light-path with different types of loops along it.

5 PATH REALIZATIONS FOR A MULTIPLY-SCATTERED, TRANSMITTED LIGHT-RAY

Since the various loops along a light-path can generally be performed in more than one sequence, the next task is to find the number of possible realizations of a light-path along which the various types of loops take place a given number of times. Starting point is to observe that all possible detour paths in the multi-layer system with only a single type of loop l_{pq}, are generated by the expression

$$\delta_N(l_{pq}) = (1 - l_{pq})^{-1}, \tag{23}$$

which gives the well-known geometric series that also appears in the transmission coefficient of for instance the Fabry-Perot interferometer [1]. In this case, when only one type of loop is included, there is only one realization for the path of a transmitted light-ray with a given number of loops along it. When more than one type of loop is included, there can be more than one realization for the path of the transmitted light-ray with a given number of loops along it, because permutations of different types of loops can result in new paths.

Although the loops are the basis path elements, along the actual path these loops are not necessarily fully completed one after another, see for instance the original path that belongs to loops l_{13} and l_{12} in Figure 2, where l_{12} has already started before l_{13} has been completed. The latter is finished only after the full performance of l_{12}, so it is as if l_{12} takes place 'within' l_{13}, the loops are nested and in order to be able to speak about a sequence of loops, we have to be more specific. Every loop l_{pq} starts with a reflection at interface q and we say that it is performed, though it is not yet completed, at the moment that the opposite reflection has taken place, at interface p. Within this agreement, l_{13} takes place before l_{12} along the example path from Figure 2.

Consider a continuous path of a transmitted light-ray with two different types of loops on it, type l_{pq} and type $l_{p'q'}$, with $(p,q) \neq (p',q')$. We also put, without loss of generality, $p \leq p'$. Since every back-forth reflection must be initiated by a rightwards propagating light-ray that impinges upon the rightmost interface of the loop, l_{pq} can be followed by $l_{p'q'}$ only if $p < q'$. This requirement is always fulfilled because $p \leq p' < q'$. The reverse order, $l_{p'q'}$ followed by l_{pq}, can only take place if $p' < q$ which means that the loops should be located in spatially partly overlapping layers. So if $p' < q$, there are $\binom{n}{m}$ realizations for the path of a light-ray which performs m loops of the one type and $n-m$ of the other whereas there is only one realization if $p' \geq q$. Therefore, the proper sum of amplitude coefficients of all detour paths that result from solely these two loop types is obtained from the expression

$$\delta_N\left(l_{pq}, l_{p'q'}\right) = \left[1 - \left(l_{pq} + l_{p'q'}\right)\right]^{-1} \quad \text{if} \quad p' < q, \quad (24a)$$

$$\delta_N\left(l_{pq}, l_{p'q'}\right) = \left(1 - l_{pq}\right)^{-1}\left(1 - l_{p'q'}\right)^{-1} \quad \text{if} \quad p' \geq q, \quad (24b)$$

as can be immediately verified by working out the terms in the generated series. When the multiplication in Eq. (24b) is carried out, this equation reads as $\delta_N\left(l_{pq}, l_{p'q'}\right)\Big|_{p'\geq q} = \left[1 - \left(l_{pq} + l_{p'q'} - l_{pq}l_{p'q'}\right)\right]^{-1}$. The bilinear term in the denominator of this expression effects subtraction of those paths that have the forbidden loop sequence involving type l_{pq} and $l_{p'q'}$. This completes the set of rules for combining loops of different types such that the correct number of realizations of the paths follows. In the next section, these rules will be applied to all loops present in the multi-layer medium, resulting in the expression for the system's transmission coefficient.

6 TRANSMISSION COEFFICIENT VIA SUM OF ALL POSSIBLE PATHS

In this section, we derive a recurrent relation for the factor δ_N in the transmission coefficient of Eq. (19), the factor that results from the sum of the amplitude coefficients of all possible detour paths of the light-rays in the multi-layer medium. In the trivial case of having zero layers, there is only one interface between subspace L and R. Eq. (22) gives that in this system, the light-path cannot perform any loops. The amplitude of the transmitted field is therefore unaffected by the detour path-part δ_0 of the transmission coefficient,

$$\delta_0 = 1. \quad (25)$$

In the case of a single layer, Eq. (22) gives the single loop between the entrance and exit interface, l_{12}. The sum over all loops l_{12} is obtained from Eq. (23) as

$$\delta_1 = \left(1 - l_{12}\right)^{-1}. \quad (26)$$

For the double-layer system, Eq. (22) gives the three loops l_{12}, l_{13} and l_{23}. The sum of all possible allowed combinations of these three loops can be obtained in two ways. The first way is to start with l_{12} and l_{23} and leave out l_{13}. According to Eq. (24), $\delta_2\left(l_{12}, l_{23}\right) = \left(1 - \left(l_{12} + l_{23}\left(1 - l_{12}\right)\right)\right)^{-1}$. Now l_{13} should be added as a loop that is located in spatially partly overlapping layers with both l_{12} and l_{23}. This gives, with Eq. (24), that the sum generating expression for the two-layer system with all loops present is equal to

$$\delta_2 = \left(1 - \left(l_{12} + l_{23}\left(1 - l_{12}\right) + l_{13}\right)\right)^{-1}. \quad (27)$$

The other way is to start with the spatially partly overlapping loops l_{12} and l_{13} in the absence of l_{23}. According to Eq. (24), $\delta_2(l_{12}, l_{13}) = \left(1 - \left(l_{12} + l_{13}\right)\right)^{-1}$. Now l_{23} should be included as a loop that has no spatial overlap with l_{12} but as a loop that does have spatial overlap with l_{13}. This also results in Eq. (27). Similarly as in the case of a one-layer medium, where one type of loop is generated to all orders by Eq. (26), the expression that generates the three types of loops to all orders in a two-layer medium, Eq. (27), generates a geometric series as well, but now this series has the argument

$$L_2 = l_{12} + \left(1 - l_{12}\right) l_{23} + l_{13}. \quad (28)$$

This means that, regarding its transmission coefficient, the two-layer medium with all different types of loops can be described as a single-layer medium with one single type of loop with corresponding amplitude coefficient given by Eq. (28). From layer-by-layer addition with repeated application of the result that every two-layer system can be represented by an equivalent one-layer system with only one type of loop present in it, it follows that the detour path-factor of the N-layer system has the form

$$\delta_N = \left(1 - L_N\right)^{-1}, \quad (29)$$

where L_N is the amplitude coefficient of the single loop in the equivalent one-layer system. We will derive a recurrent relation for L_N for $N = 1, 2, \ldots$. From Eqs. (25) and (29) it follows that

$$L_0 = 0. \quad (30)$$

Under the addition of an N-th layer to a system with $N-1$ layers, the N new loops $l_{1,N+1}$ to $l_{N,N+1}$ emerge. Of these, $l_{1,N+1}$ has partial spatial overlap with all other loops, hence they should be combined as in Eq. (24), giving $L_N\left(L_{N-1}, l_{1,N+1}\right) = L_{N-1} + l_{1,N+1}$. Loop $l_{2,N+1}$ has partial spatial overlap with all loops except for the loops represented in L_1, therefore $L_N(L_{N-1}, l_{1,N+1}, l_{2,N+1}) = L_{N-1} + l_{1,N+1} + l_{2,N+1}(1 - L_1)$. From applying this up to and including the last new loop $l_{N,N+1}$, it follows that

$$L_N = L_{N-1} + \sum_{m=1}^{N} (1 - L_{m-1}) l_{m,N+1}. \tag{31}$$

From this equation immediately follows the recurrence relation for the transmission coefficient for the N-layer system. Summarized, the transmission coefficient is given by

$$t_N = t_N^{(0)} (1 - L_N)^{-1}, \tag{32}$$

where $t_N^{(0)}$, given by Eq. (18), is the amplitude coefficient of the direct path, which is shared by all transmitted light-rays and where L_N is the amplitude coefficient of the equivalent single-layer system loop, given by Eq. (31). Note that the effective loop L_N is multi-linear in all the elementary loops in $\{l_{pq}\}_N$.

7 CONCLUSION

We have calculated the transmission coefficient of the N-layer medium by summing the amplitude coefficients that belong to all individual transmitted light-rays. It has turned out that, just as in the case of a single-layer medium, the sum of all transmitted light-rays in the N-layer medium can be captured in a geometrical series. The basic elements in this series are back-forth reflections, or loops, between the various pairs of interfaces of the medium.

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A PRELIMINARIES

Electromagnetic fields are governed by Maxwell's equations. In a medium that does not contain free electric charges and currents, these equations read as

$$\nabla \times \mathbf{E} + \dot{\mathbf{B}} = 0, \tag{A.1a}$$
$$\nabla \times \mathbf{H} - \dot{\mathbf{D}} = 0, \tag{A.1b}$$
$$\nabla \cdot \mathbf{D} = 0, \tag{A.1c}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{A.1d}$$

where $\mathbf{E}$ is the electric, $\mathbf{H}$ the magnetic field and where $\mathbf{D} = \mathbf{D}[\mathbf{E}, \mathbf{B}]$ and $\mathbf{H} = \mathbf{H}[\mathbf{E}, \mathbf{B}]$. In Fourier representation, with $\mathbf{F} \in \{\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}\}$,

$$\mathbf{F}(t, \mathbf{r}) = \int d\omega\, \widetilde{\mathbf{F}}(\omega; \mathbf{r}) \exp(-i\omega t), \tag{A.2}$$

where

$$\widetilde{\mathbf{F}}(\omega; \mathbf{r}) = \frac{1}{2\pi} \int dt\, \mathbf{F}(t, \mathbf{r}) \exp(i\omega t). \tag{A.3}$$

For linear and isotropic media,

$$\widetilde{\mathbf{D}} = \epsilon \widetilde{\mathbf{E}}, \tag{A.4a}$$
$$\widetilde{\mathbf{H}} = \mu^{-1} \widetilde{\mathbf{B}}, \tag{A.4b}$$

where ϵ and μ are respectively the absolute permittivity and absolute permeability, which can both be complex functions of ω. With Eqs. (A.4), Eqs. (A.1) lead to

$$\left(\nabla^2 + \mathbf{k}^2\right) \widetilde{\mathbf{E}} + (\nabla \ln \mu) \times \nabla \times \widetilde{\mathbf{E}} + \nabla \left(\widetilde{\mathbf{E}} \cdot \nabla \ln \epsilon\right) = 0, \tag{A.5a}$$

$$\left(\nabla^2 + \mathbf{k}^2\right) \widetilde{\mathbf{H}} + (\nabla \ln \epsilon) \times \nabla \times \widetilde{\mathbf{H}} + \nabla \left(\widetilde{\mathbf{H}} \cdot \nabla \ln \mu\right) = 0, \tag{A.5b}$$

where

$$\mathbf{k}^2 = \omega^2 \epsilon \mu. \tag{A.6}$$

In the homogeneous subspaces q of Section 2, with permittivities ϵ_q and permeabilities μ_q, the gradient terms in Eqs. (A.5) vanish and the fields satisfy Helmholtz' equation,

$$\left(\nabla^2 + \mathbf{k}_q^2\right) \left\{ \begin{array}{c} \widetilde{\mathbf{E}}_q \\ \widetilde{\mathbf{H}}_q \end{array} \right\} = 0, \tag{A.7}$$

where now $\widetilde{\mathbf{E}}_q$ and $\widetilde{\mathbf{H}}_q$ denote the Fourier transformed fields in subspace q and $\mathbf{k}_q^2 = \omega^2 \epsilon_q \mu_q$. Eq. (A.7) forms the starting point of Section 3.

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