Journal of the European Optical Society - Rapid Publications **5**, 10022 (2010) www.jeos.org

The Legendre transformations in Hamiltonian optics

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The Legendre transformations are an important tool in theoretical physics. They play a critical role in mechanics, optics, and thermodynamics. In Hamiltonian optics the Legendre transformations appear twice: as the connection between the Lagrangian and the Hamiltonian and as relations among eikonals. In this article interconnections between these two types of Legendre transformations have been investigated. Using the method of "transition to the centre and difference coordinates" it is shown that four Legendre transformations which connect point, point-angle, angle-point, and angle eikonals of an optical system correspond to four Legendre transformations which connect four systems of equations: Euler's equations, Hamilton's equations, and two unknown before pairs of equations. [DOI: 10.2971/jeos.2010.10022]

Keywords: Legendre transformations, eikonal theory, Euler's equations, Hamilton's equations.

1 INTRODUCTION

The best way to solve a problem is to look at it from the right point of view. So it is very important for everybody to have a possibility to see the problem from various points of view. In physics this possibility is given by one-to-one correspondences, for example the Fourier transformations in Fourier-optics and the Legendre transformations in Hamiltonian optics. Investigations of intrinsic one-to-one transformations are very useful for understanding the structure of the theory and very helpful for solving various practical problems.

Note that in Hamiltonian optics the Legendre transformations appear twice: as a connection between the Lagrangian and the Hamiltonian and as relations among eikonals. Traditionally these types of the Legendre transformations are used independently. In this article we show interconnections between them.

2 FERMAT'S PRINCIPLE AS THE BASIS OF HAMILTONIAN OPTICS

Let us consider a layer of an inhomogeneous optical medium with the refractive index distribution $n(x,y,z)$ which is restricted by an input plane $\{(x,y)\} = \{(x,y,z) : z = 0\}$ and an output plane $\{(x',y') = \{(x,y,z) : z = Z\}$. A collection of virtual paths connecting an input point (x,y) with an output point (x',y') can be labelled with a parameter u. Let all these paths γ_u be projected onto the z-axis so that they can be described by functions $x(z)$ and $y(z)$. Hence, the entire set of paths running from the input point (x,y) to the output point (x',y') is given by two functions of two parameters: $\gamma_u(z) = \{x(z,u), y(z,u)\}$ (see Figure 1). Following Luneburg [1], we use the z-axis as the independent variable, similar to the time axis in analytical mechanics.

The tangent to a path $\gamma_u(z)$ at a given point P^* (see Figure 1)

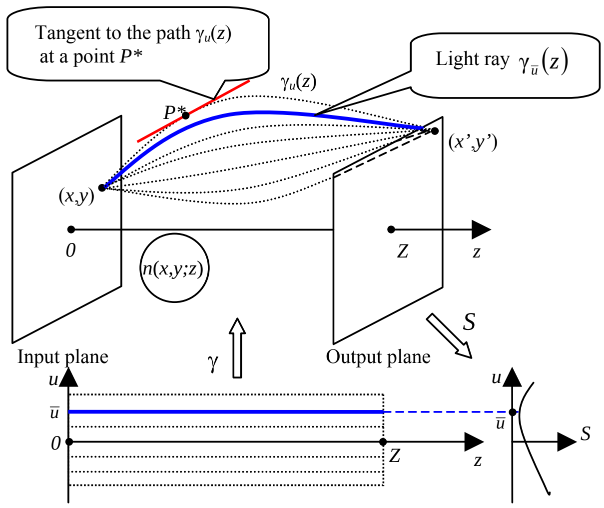

FIG. 1 Illustration of Fermat's principle applied to the layer of an inhomogeneous medium. The Fermat's principle contains three statements: 1) a collection of virtual paths connecting an input point (x,y) with an output point (x',y') can be labelled with a parameter u: $\gamma : (z,u) \to (x,y;z) \in R^3$ i.e. $\gamma_u(0) = (x,y;0) = (x,y)$ and $\gamma_u(Z) = (x,y;Z) = (x',y')$; 2) all virtual paths going through the optical inhomogeneous medium $n(x,y,z)$ have the optical length S: $u \to \int_{\gamma_u} n(x,y,z)\sqrt{1+\dot{x}^2+\dot{y}^2}dz$; and 3) light travels from the input point (x,y) to the output point (x',y') along a path for which the optical path is stationary with respect to all possible neighbouring paths, i.e. $\frac{\partial S}{\partial u}\Big|_{u=\bar{u}} = 0$.

can be specified by angles (θ,φ) of a spherical coordinate system $(\varphi \in [0,2\pi), \theta \in (0,\pi/2])$, but it is more convenient to use "velocities" for this purpose (see Figure 2(a)):

$$\dot{x} = \frac{dx}{dz} = \tan\theta \cdot \sin\varphi\,, \tag{1a}$$

$$\dot{y} = \frac{dy}{dz} = \tan\theta \cdot \cos\varphi\,. \tag{1b}$$

Received 25 November, 2009; published 27 April, 2010 ISSN 1990-2573

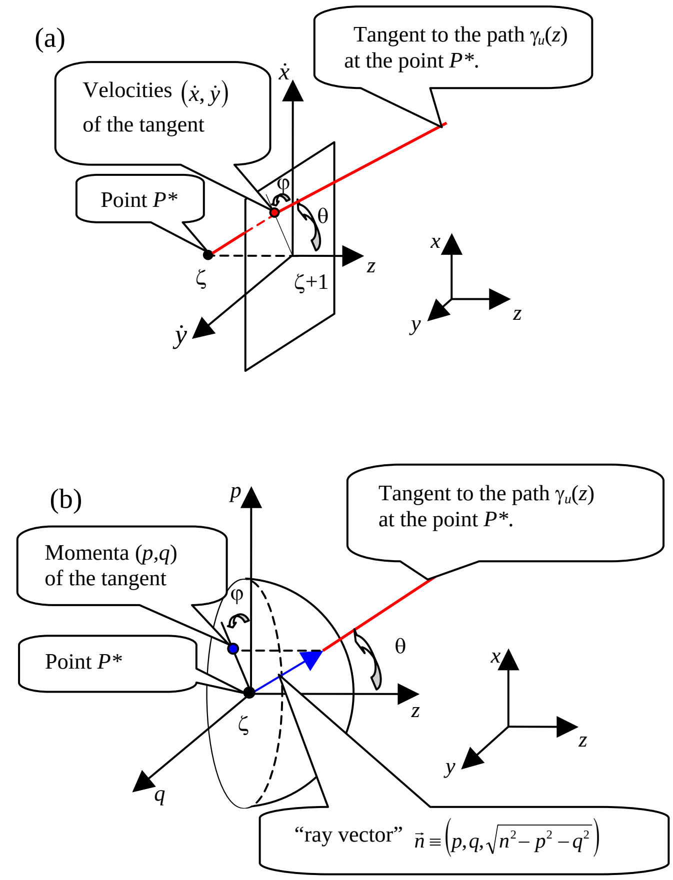

FIG. 2 Direction of the tangent to a path $\gamma_u(z)$ at a given point P^* $(\gamma_u(\zeta), \zeta)$ can be characterized by angles (θ, φ) of the spherical coordinate system ($\varphi \in (0, 2\pi)$, $\theta \in (0, \pi/2)$), by (a) "velocities" or by (b) momenta.

It takes time $t(u)$ for the light to travel along the path $\gamma_u(z)$ from (x, y) to (x', y'). This value is given by

$$\begin{aligned} t(u) &= \int_{\gamma_u} \frac{\sqrt{1+\dot{x}^2+\dot{y}^2}}{v(x,y,z)} dz \\ &= \frac{1}{c} \int_{\gamma_u} n(x,y,z) \sqrt{1+\dot{x}^2+\dot{y}^2} dz \\ &= \frac{1}{c} S(x,y;x'y';u) . \end{aligned} \tag{2}$$

The integral of the product of the refractive index and the geometrical path length $S(x,y;x'y';u) = \int_{\gamma_u} L(x,y;\dot{x},\dot{y};z)\, dz$ is called the optical length of the path $\gamma_u(z)$ from (x, y) to (x', y'). The integrand

$$L(x,y;\dot{x},\dot{y};z) = n(x,y,z) \sqrt{1+\dot{x}^2+\dot{y}^2} \tag{3}$$

is the Lagrangian.

Hamiltonian optics is based on the concept that light travels along the path, which satisfies Fermat's principle. It states that the light travels from one point (x, y) to another (x, y) along the path $\gamma_u(z)$ for which the travel time is stationary with respect to all possible neighbouring paths. Because the speed of light in vacuum is constant, according to Eq. (2), Fermat's principle can be expressed in terms of the optical path as well as the transit time [1]– [3]:

$$\left.\frac{\partial S(x,y;x',y';u)}{\partial u}\right|_{u=\bar{u}} = 0 \tag{4}$$

for any arbitrary collections of paths (see Figure 1).

The variation of the path length $S(x, y; x'y'; u)$ with respect to u is described by [2]

$$\begin{aligned} \frac{\partial S(x,y;x',y';u)}{\partial u} &= \frac{\partial}{\partial u} \int_{\gamma_u} L(x,y;\dot{x},\dot{y};z)\, dz \\ &= \int_{\gamma_u} \left(\frac{\partial L}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial u} \right) dz + (y) \\ &= \int_{\gamma_u} \frac{\partial L}{\partial x}\frac{\partial x}{\partial u} dz + \int_{\gamma_u} \frac{\partial L}{\partial \dot{x}} \frac{d}{dz}\left(\frac{\partial x}{\partial u}\right) dz + (y) \\ &= \left.\frac{\partial L}{\partial \dot{x}}\frac{\partial x}{\partial u}\right|_0^Z + \int_{\gamma_u} \left[\frac{\partial L}{\partial x} - \frac{d}{dz}\left(\frac{\partial L}{\partial \dot{x}}\right)\right] \frac{\partial x}{\partial u} dz + (y) \end{aligned} \tag{5}$$

The formulas are written for x-terms only. They can be completed by similar terms in y as indicated.

3 THE LEGENDRE TRANSFORMATIONS BETWEEN LAGRANGIAN AND HAMILTONIAN

We choose varied paths between two points (x, y) and (x', y'). Eq. (5) shows that to satisfy Fermat's principle, Eq. (4), we must have

$$\frac{\partial L}{\partial x} - \frac{d}{dz}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \text{ and } \frac{\partial L}{\partial y} - \frac{d}{dz}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \tag{6}$$

for the ray in the inhomogeneous medium. They are the Euler's equations [2] that determine the path of a ray in an inhomogeneous medium.

We can rewrite Eq. (6) in the form

$$\dot{p} = \frac{\partial L}{\partial x}, p \equiv \frac{\partial L}{\partial \dot{x}} \text{ and } \dot{q} = \frac{\partial L}{\partial y}, q \equiv \frac{\partial L}{\partial \dot{y}} \tag{7}$$

Thus, we have two pairs of differential equations. The variables $p \equiv \frac{\partial L}{\partial \dot{x}}$ and $q \equiv \frac{\partial L}{\partial \dot{y}}$ are called "momenta". The momenta as well as "velocities" specify the direction of a ray in space. In the case of the optical Lagrangian, Eq. (3), the momenta are

$$p \equiv \frac{\partial L}{\partial \dot{x}} = n(x,y,z) \frac{\dot{x}}{\sqrt{1+\dot{x}^2+\dot{y}^2}} = n \cdot \sin\theta \cdot \sin\varphi \tag{8a}$$

$$q \equiv \frac{\partial L}{\partial \dot{y}} = n(x,y,z) \frac{\dot{y}}{\sqrt{1+\dot{x}^2+\dot{y}^2}} = n \cdot \sin\theta \cdot \cos\varphi \tag{8b}$$

where ($\varphi \in [0, 2\pi), \theta \in (0, \pi/2]$) (see Figure 2(b)). The quantities p and q can be interpreted as "optical direction cosines" [1]. They are two independent components of the "ray vector" $\vec{n} \equiv (p, q, \sqrt{n^2 - p^2 - q^2})$ [3], which is parallel to the tangent to the path at the point P^a* (see Figure 1).

It is known that the Legendre transformation creates a new function which contains the same information as the old, but it appears to be the function of different variables. There are a few different definitions of the Legendre transformation [5]. Let us consider the direct and inverse Legendre transformations with signs that are used in the theory of eikonals [1]– [3,6]– [8] (see below).

The direct Legendre transformation $\mathcal{L}_{x\to p,y\to q}\{...\}$ of a real-valued differentiable function $f(x,y)$ is the real-valued differentiable function $g(p,q)$ of a new variable $p \equiv \frac{\partial f}{\partial x}$, $q \equiv \frac{\partial f}{\partial y}$, which is defined as [3,5]– [7]

$$g(p,q) = \mathcal{L}_{x\to p,y\to q}\{f(x,y)\} \equiv f(x,y) - xp - yq\,. \tag{9a}$$

The inverse Legendre transformation $\mathcal{L}_{x\to p,y\to q}\{...\}$ of a real-valued differentiable function $g(p,q)$ is the real-valued differentiable function $f(x,y)$ of a new variable $x \equiv \frac{\partial g}{\partial p}$, $y \equiv \frac{\partial g}{\partial q}$, which is defined as [3,5]– [7]

$$f(x,y) = \mathcal{L}^{-1}_{p\to x,q\to y}\{g(p,q)\} \equiv g(p,q) + xp + yq\,. \tag{9b}$$

If $f(x,y)$ is a real-valued differentiable and regular function [4, 5] $\left(\det\begin{pmatrix} f^{||}_{xx} & f^{||}_{xy} \\ f^{||}_{yx} & f^{||}_{yy}\end{pmatrix} \neq 0\right)$ then the composition of the direct Legendre transformation and the inverse Legendre transformation (or the composition of the inverse Legendre transformation and the direct Legendre transformation) is the identity operator.

Note that the Lagrangian, Eq. (3), is a real-valued continuously differentiable and regular function of variables $\dot{x}$ and $\dot{y}$. Thus, we can apply to it the direct Legendre transformation on these variables. The direct Legendre transformation of the Lagrangian $L(x,y;\dot{x},\dot{y};z)$ with a minus sign is called the Hamiltonian $H(p,q;x,y;z)$:

$$H(p,q;x,y;z) = -\mathcal{L}_{\dot{x}\to p,\dot{y}\to q}\{L(x,y;\dot{x},\dot{y};z)\}\,. \tag{10}$$

Substituting definitions Eq. (3) and Eqs. (8) into Eq. (10), we can see that the Hamiltonian, Eq. (10), takes the form [8]:

$$\begin{aligned} H(p,q;x,y;z) &= -\left[L(x,y;\dot{x},\dot{y};z) - \dot{x}p - \dot{y}q\right] \\ &= -n(x,y;z)\left(\sqrt{1+\dot{x}^2+v\dot{y}^2} - \frac{\dot{x}^2}{\sqrt{1+\dot{x}^2+\dot{y}^2}} - \frac{\dot{y}^2}{\sqrt{1+\dot{x}^2+\dot{y}^2}}\right) \\ &= -\sqrt{n^2(x,y;z) - p^2 - q^2} \\ &= -n(x,y;z)\cos\theta(x,y;z) \end{aligned} \tag{11}$$

since

$$\frac{n^2}{1+\dot{x}^2+\dot{y}^2} = n^2 - p^2 - q^2\,. \tag{12}$$

The geometrical interpretation of the Legendre transformations [4, 7, 9, 10], which connects the Lagrangian, $L(x;\dot{x};z) = n(x,z)\sqrt{1+\dot{x}^2}$, and the Hamiltonian, $H(p;x;z) = -\sqrt{n^2(x;z) - p^2}$, in the meridional plane is shown in Figure 3.

The total differential of the Hamiltonian

$$dH(p,q;x,y;z) = \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy + \frac{\partial H}{\partial z}dz \tag{13}$$

is equal to the total differential of $H = \dot{x}p + \dot{y}q - L$ for $p \equiv \frac{\partial L}{\partial \dot{x}}, q \equiv \frac{\partial L}{\partial \dot{y}}$,

$$dH(p,q;x,y;z) = \dot{x}dp + \dot{y}dq - \frac{\partial L}{\partial x}dx - \frac{\partial L}{\partial y}dy - \frac{\partial L}{\partial z}dz\,. \tag{14}$$

Both expressions for dH must be the same. Therefore

$$\dot{x} = \frac{\partial H}{\partial p},\ \dot{y} = \frac{\partial H}{\partial q},\ \frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x},\ \frac{\partial H}{\partial y} = -\frac{\partial L}{\partial y},\ \frac{\partial H}{\partial z} = -\frac{\partial L}{\partial z} \tag{15}$$

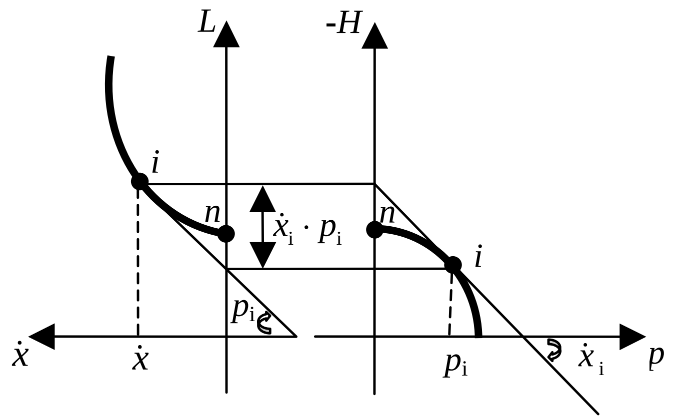

FIG. 3 The geometrical interpretation of the Legendre transformations between the Lagrangian, $L(x;\dot{x};z) = (nx,z)\sqrt{1+\dot{x}^2}$, and the Hamiltonian, $H(p;x;z) = -\sqrt{n^2(x;z) - p^2}$, in the meridional plane. The direct Legendre transformation of the function, $L(\dot{x})$ into the function $-H_i(p_i)$. Let $\dot{x}_i$ be a given number. We draw the tangent line to the graph of $L(\dot{x})$ at the point $\dot{x}_i$. The tangent line has the slope $p_i \equiv \frac{\partial L}{\partial \dot{x}}\Big|_{\dot{x}=\dot{x}_i}$ and intersects the L-axis at the point $(0,-H_i)$. The inverse Legendre transformation of the function $-H_i(p_i)$ into the function $L(\dot{x})$. Let p_i be a given number. We draw the tangent line to the graph of $-H(p)$ at the point p_i. The tangent line has the slope $\dot{x}_i \equiv -\frac{\partial H}{\partial p}\Big|_{p=p_i}$ and intersects the H-axis at the point $(0, L_i)$. The vertical distance between the horizontal lines is the product $\dot{x}_i p_i$. By subtracting this product from p_i we obtain x_i or by subtracting the product from x_i we obtain p_i.

Applying Euler's equations, Eqs. (8), we find that light rays are the solutions of Hamilton's equations [4]

$$\dot{p} = -\frac{\partial H}{\partial x},\ \dot{x} = \frac{\partial H}{\partial p} \text{ and } \dot{q} = -\frac{\partial H}{\partial y},\ \dot{y} = \frac{\partial H}{\partial q}\,. \tag{16}$$

4 LEGENDRE TRANSFORMATIONS IN THE EIKONAL THEORY

Let us assume that the chosen path satisfies the Euler-Lagrange equations. Then this path is the legitimate light ray, i.e. it satisfies Fermat's principle, Eq. (6). Substituting Eq. (7) and Eqs. (8) into Eq. (5), we obtain

$$\frac{\partial S}{\partial u} = p\frac{\partial x}{\partial u}\Big|_0^Z + q\frac{\partial y}{\partial u}\Big|_0^Z\,. \tag{17}$$

The optical length S of the ray between the points $P(x,y)$ and $P'(x',y')$ (see Figure 4), considered as a function of their four coordinates $(x,y;x',y')$, is called the "point eikonal".

Therefore we can rewrite Eq. (14) as

$$dS = p'dx' + q'dx' - pdx - qdx\,. \tag{18}$$

Variables with primes refer to the output plane.

Instead of the point eikonal, S, we can also use the point-angle eikonal, V, angle-point eikonal, V', or angle eikonal, T [1]–[3,6]– [8], which are related to the point eikonal S by the Leg-

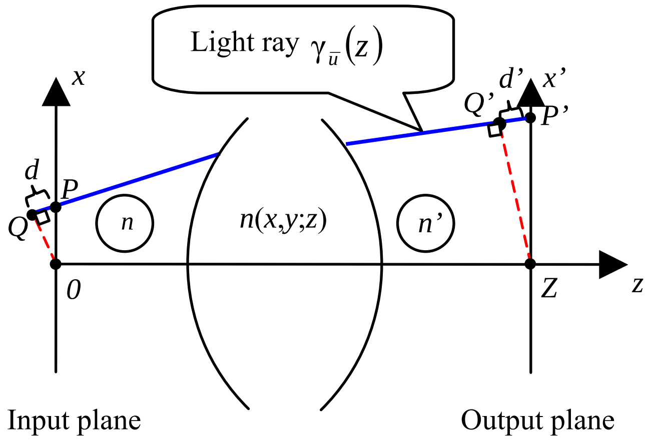

FIG. 4 Geometrical interpretation of the eikonals when the initial and the final media are homogeneous (in meridional plane): the point eikonal S is the optical length of the ray from P to P'; the point-angle eikonal V is the optical length of the ray from P to Q'; the angle-point eikonal V' is the optical length of the ray from Q to P'; and the angle eikonal T is the optical length of the ray from Q to Q'.

endre transformations

$$\begin{aligned} V(x,y;p',q') &= \mathcal{L}_{x'\to p',y'\to q'}\{S(x,y;x',y')\} \\ &\equiv S(x,y;x',y') - x'p' - y'q' \,, \qquad (19a) \\ V'(p,q;x',y') &= \mathcal{L}^{-1}_{x\to p,y\to q}\{S(x,y;x',y')\} \\ &\equiv S(x,y;x',y') + xp + yq \,, \qquad (19b) \\ T(p,q;p',q') &= \mathcal{L}_{x'\to p',y'\to q'}\mathcal{L}^{-1}_{x\to p,y\to q}\{S(x,y;x'y')\} \\ &\equiv S(x,y;x',y') - x'p' - y'q' + xp + yq \,. \qquad (19c) \end{aligned}$$

The eikonals, Eqs. (19), possess important properties

$$p' = \frac{\partial S}{\partial x'}, \quad q' = \frac{\partial S}{\partial y'}, \quad p = -\frac{\partial S}{\partial x}, \quad q = -\frac{\partial S}{\partial y} \qquad (20a)$$

$$x' = -\frac{\partial V}{\partial p'}, \quad y' = -\frac{\partial V}{\partial q'}, \quad p = -\frac{\partial V}{\partial x}, \quad q = -\frac{\partial V}{\partial y} \qquad (20b)$$

$$p' = \frac{\partial V'}{\partial x'}, \quad q = \frac{\partial V'}{\partial y'}, \quad x = \frac{\partial V'}{\partial p}, \quad y = \frac{\partial V'}{\partial q} \qquad (20c)$$

$$x' = -\frac{\partial T}{\partial p'}, \quad y' = -\frac{\partial T}{\partial q'}, \quad x = \frac{\partial T}{\partial p}, \quad y = \frac{\partial T}{\partial q} \qquad (20d)$$

which make it possible to compute the missing coordinates or momenta needed for complete description of the light ray.

If the input plane is situated in a homogeneous medium with the refractive index n, called "object space", and the output plane is situated in a homogeneous medium with the refractive index n', called "image space", then we can consider the layer of an inhomogeneous optical medium as an optical system (see Figure 4). In the meridional plane x-z of this system the terms xp and $x'p'$ have a simple geometrical interpretation $xp = nd$ and $x'p' = n'd'$ where d is the distance along the ray from the point $P(x,0)$ to the foot Q of the perpendicular drawn from the origin 0 to the ray at P and d' is the distance along the ray from the point $P'(x',0)$ to the foot Q' of the perpendicular drawn from the point Z to the ray at P' (see Figure 4) [3,6,7].

The Legendre transformations are important. There are many different kinds of optical systems, and one kind of eikonal is not sufficient to describe all of them. Here are some examples [11]. A point characteristic cannot be used when the second plane is the image plane. A mixed characteristic cannot be used when one plane is coincident with the focal plane. An angle characteristic cannot be used when dealing with afocal systems.

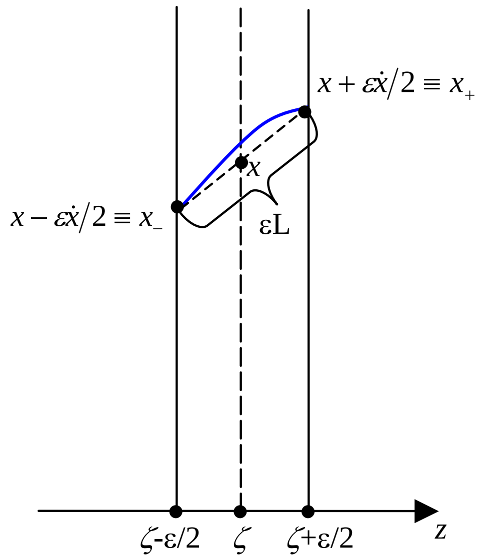

FIG. 5 The point eikonal of a thin layer of an optically inhomogeneous medium restricted by the input $\{(x_-,y_-)\} \equiv \{(x,y;z) : z = \zeta - \varepsilon/2\}$ and the output $\{(x_+y_+)\} \equiv \{(x,y;z) : z = \zeta + \varepsilon/2\}$ planes (in meridional cross section).

5 THE CONNECTION BETWEEN TWO TYPES OF LEGENDRE TRANSFORMATIONS

We show that the differential equations describing the behaviour of the light ray at each point of the optical inhomogeneous medium can be derived from the differential properties of eikonals.

In [7] it is noted that the relationships that connect the eikonals of an optical system are valid for any position of the input $\{(x_-,y_-)\} \equiv \{(x,y;z) : z = \zeta - \varepsilon/2\}$ and output $\{(x_+,y_+)\} \equiv \{(x,y;z) : z = \zeta + \varepsilon/2\}$ planes (see Figure 5). Then it is possible to transform the four eikonals to four systems of differential equations of a light ray.

Let the input (x_-,y_-) and output (x_+,y_+) planes be separated by a layer of a medium of thickness ε that is less than the characteristic size of the optical inhomogeneity. Therefore in this case it is convenient to express Eq. (19c) in terms of the centre and difference coordinates [7,12]– [14]:

$$x \equiv \frac{x_+ + x_-}{2}, y \equiv \frac{y_+ + y_-}{2}, \quad \dot{x} \equiv \frac{x_+ - x_-}{\varepsilon}, \dot{y} \equiv \frac{y_+ - y_-}{\varepsilon}, \qquad (21a)$$

$$p \equiv \frac{p_+ + p_-}{2}, q \equiv \frac{q_+ + q_-}{2}, \quad \dot{p} \equiv \frac{p_+ - p_-}{\varepsilon}, \dot{q} \equiv \frac{q_+ - q_-}{\varepsilon}. \qquad (21b)$$

Using these coordinates, we obtain

$$\begin{aligned}&T\left(p-\varepsilon\frac{\dot p}{2},q-\varepsilon\frac{\dot q}{2};p+\varepsilon\frac{\dot p}{2},q+\varepsilon\frac{\dot q}{2}\right)\\&\equiv S\left(x-\varepsilon\frac{\dot x}{2},y-\varepsilon\frac{\dot y}{2};x+\varepsilon\frac{\dot x}{2},y+\varepsilon\frac{\dot y}{2}\right)-\left(p+\varepsilon\frac{\dot p}{2}\right)\left(x+\varepsilon\frac{\dot x}{2}\right)\\&-\left(q+\varepsilon\frac{\dot q}{2}\right)\left(y+\varepsilon\frac{\dot y}{2}\right)+\left(p-\varepsilon\frac{\dot p}{2}\right)\left(x-\varepsilon\frac{\dot x}{2}\right)+\left(q-\varepsilon\frac{\dot q}{2}\right)\left(y-\varepsilon\frac{\dot y}{2}\right)\\&=S\left(x-\varepsilon\frac{\dot x}{2},y-\varepsilon\frac{\dot y}{2};x+\varepsilon\frac{\dot x}{2},y+\varepsilon\frac{\dot y}{2}\right)-\varepsilon\left(\dot p x+\dot q y+p\dot x+q\dot y\right).\end{aligned} \tag{22}$$

Note that the point eikonal $S\left(x-\varepsilon\frac{\dot x}{2},y-\varepsilon\frac{\dot y}{2};x+\varepsilon\frac{\dot x}{2},y+\varepsilon\frac{\dot y}{2}\right)$ and angle eikonal $T\left(p-\varepsilon\frac{\dot p}{2},q-\varepsilon\frac{\dot q}{2};p+\varepsilon\frac{\dot p}{2},q+\varepsilon\frac{\dot q}{2}\right)$ of an infinitely thin layer are proportional to its thickness ε. This makes it possible to introduce the corresponding specific eikonals

$$L\left(x,y;\dot x,\dot y;z\right)\equiv\frac{1}{\varepsilon}S\left(x-\varepsilon\frac{\dot x}{2},y-\varepsilon\frac{\dot y}{2};x+\varepsilon\frac{\dot x}{2},y+\varepsilon\frac{\dot y}{2}\right) \tag{23a}$$

and

$$M\left(\dot p,\dot q;p,q;z\right)\equiv\frac{1}{\varepsilon}T\left(p-\varepsilon\frac{\dot p}{2},q-\varepsilon\frac{\dot q}{2};p+\varepsilon\frac{\dot p}{2},q+\varepsilon\frac{\dot q}{2}\right). \tag{23b}$$

In this notation, the Legendre transformation of Eq. (22) can be written more compactly as

$$\begin{aligned}M\left(\dot p,\dot q;p,q;z\right)&=-\mathcal{L}_{\dot x\to p,\dot y\to q}\mathcal{L}_{x\to\dot p y\to\dot q}\left\{L\left(x,y;\dot x,\dot y;z\right)\right\}\\&=-\left[x\dot p+y\dot q+p\dot x+q\dot y-L\left(x,y;\dot x,\dot y;z\right)\right]\end{aligned}. \tag{24a}$$

Note that Eq. (19a) provides a basis for introducing two more specific eikonals by means of Legendre transformations:

$$\begin{aligned}H\left(p,q;x,y;z\right)&\equiv-\mathcal{L}_{\dot x\to p,\dot y\to q}\left\{L\left(x,y;\dot x,\dot y;z\right)\right\}\\&=p\dot x+q\dot y-L\left(x,y;\dot x,\dot y;z\right)\end{aligned} \tag{24b}$$

and

$$\begin{aligned}N\left(\dot p,\dot q;\dot x,\dot y;z\right)&\equiv-L_{x\to\dot p,y\to\dot q}\left\{L\left(x,y;\dot x,\dot y;z\right)\right\}\\&=x\dot p+y\dot q-L\left(x,y;\dot x,\dot y;z\right).\end{aligned} \tag{24c}$$

The specific eikonals, Eqs. (24) possess important properties

$$p=\frac{\partial L}{\partial\dot x},\quad q=\frac{\partial L}{\partial\dot y},\quad \dot p=\frac{\partial L}{\partial x},\quad \dot q=\frac{\partial L}{\partial y} \tag{25a}$$

$$\dot x=\frac{\partial H}{\partial p},\quad \dot y=\frac{\partial H}{\partial q},\quad \dot p=-\frac{\partial H}{\partial x},\quad \dot q=-\frac{\partial H}{\partial y} \tag{25b}$$

$$p=-\frac{\partial N}{\partial\dot x},\quad q=-\frac{\partial N}{\partial\dot y},\quad x=\frac{\partial N}{\partial\dot p},\quad y=\frac{\partial N}{\partial\dot q} \tag{25c}$$

$$\dot x=-\frac{\partial M}{\partial p},\quad \dot y=-\frac{\partial M}{\partial q},\quad x=-\frac{\partial M}{\partial\dot p},\quad y=-\frac{\partial M}{\partial\dot q} \tag{25d}$$

The Lagrangian $L\left(x,y;\dot x,\dot y;z\right)$ as a function of $\dot x$ and $\dot y$ is rational, so for the direct Legendre transformation, Eq. (24b), there is an inverse Legendre transformation. If the Lagrangian $L\left(x,y;\dot x,\dot y;z\right)$ as a function of x and y is rational too, for the direct Legendre transformation, Eq. (24c), there is an inverse Legendre transformation. Thus, according to the definition of the Lagrangian, Eq. (3), the Legendre transformations Eqs. (24a) and (24c) can be useful for the gradient-index optics [15].

It is natural to identify functions $L\left(x,y;\dot x,\dot y;z\right)$ and $H(p,q;x,y;z)$ with the Lagrangian, Eq. (3), and the Hamiltonian, Eq. (10). The functions $M\left(\dot p,\dot q;p,q;z\right)$ and $N\left(\dot p,\dot q;\dot x,\dot y;z\right)$ are new. Thus, the consequences of the theory of eikonals are not only the well-known Euler's equation, Eq. (25a) and Hamilton's equation, Eq. (25b), but also new equations Eqs. (25c) and (25d) not used earlier in optics (and mechanics).

6 ANALOG OF EULER'S EQUATIONS IN MOMENTUM REPRESENTATION

The fundamental symmetry of the theory of Hamilton is the equivalence of coordinate and momentum representations [16]. We will show, how with the help of the function $M(\dot p,\dot q;p,q;z)$ it is possible to deduce the analog of Euler's equation in momentum representation (compare with Eqs. (4) and (7)).

Let us consider a scalar monochromatic wave going through the optical system. A single plane wave with direction $\vec n\equiv\left(p,q,\sqrt{n^2-p^2-q^2}\right)$ entering the system creates a set of plane waves in image space, traveling in all directions [12]. One of these plane waves has direction $\vec n'\equiv\left(p',q',\sqrt{n'^2-p'^2-q'^2}\right)$. The relation between these plane waves can be discussed more conveniently when we consider the particular wavefront with momenta (p,q) passing through the origin 0 in object space, "input wavefront", and the particular wavefront with momenta (p',q') passing through the point Z in image space, "output wavefront" (see Figure 6).

The optical length T of the path between these wavefronts (p,q) and (p',q'), can be calculated using function $M\left(\dot p,\dot q;p,q;z\right)$,

$$T\left(u\right)\equiv\int_{\gamma_u}M\left(\dot p,\dot q;p,q;z\right)dz, \tag{26}$$

its variation with respect to w is described by

$$\begin{aligned}\frac{\partial T}{\partial w}&=\frac{\partial}{\partial w}\int_{\gamma_w}M\left(\dot p,\dot q;p,q;z\right)dz\\&=\int_{\gamma_w}\left(\frac{\partial M}{\partial p}\frac{\partial p}{\partial w}+\frac{\partial M}{\partial\dot p}\frac{\partial\dot p}{\partial w}\right)dz+(q)\\&=\int_{\gamma_w}\frac{\partial M}{\partial p}\frac{\partial p}{\partial w}dz+\int_{\gamma_w}\frac{\partial M}{\partial\dot p}\frac{d}{dz}\left(\frac{\partial p}{\partial w}\right)dz+(q)\\&=\left.\frac{\partial M}{\partial\dot p}\frac{\partial q}{\partial w}\right|_0^Z+\int_{\gamma_w}\left[\frac{\partial M}{\partial p}-\frac{d}{dz}\left(\frac{\partial M}{\partial\dot p}\right)\right]\frac{\partial p}{\partial w}dz+(q).\end{aligned} \tag{27}$$

The formulas are written for p-terms only. They can be completed by similar terms in q as indicated.

The phase difference between input and output wavefronts is proportional to the optical path between these fronts measured along the ray $\gamma_{\overline{w}}(z)$ that satisfies Fermat's principle in momentum representation:

$$\left.\frac{\partial T}{\partial w}\right|_{w=\overline{w}}=0. \tag{28}$$

This path $\gamma_{\overline{w}}(z)$ is perpendicular to the input and output wavefronts [12] (see Figure 6). The optical length T of this

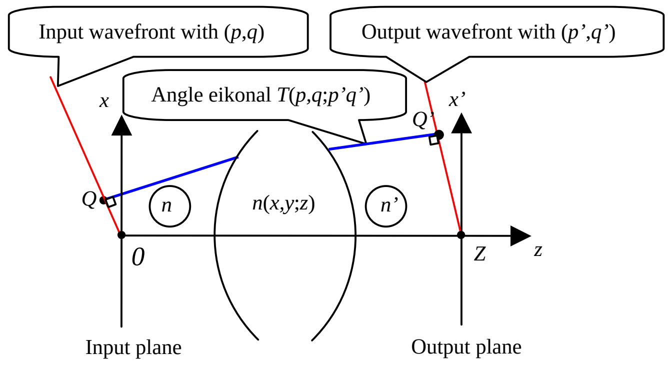

FIG. 6 The phase difference between the input wavefront with momenta (p,q) and the output wavefront with momenta (p',q') is proportional to the optical path between these wavefronts, i.e. the angle eikonal $T(p,q;p',q')$.

ray between these wavefronts (p,q) and (p',q'), considered as a function of their four coordinates $(p,q;p',q')$, is the angle eikonal (compare Figure 6 with Figure 4).

Eq. (27) shows that to satisfy Fermat's principle, Eq. (28), we must have

$$\frac{\partial M}{\partial p} - \frac{d}{dz}\left(\frac{\partial M}{\partial \dot{p}}\right) = 0 \text{ and } \frac{\partial M}{\partial q} - \frac{d}{dz}\left(\frac{\partial M}{\partial \dot{q}}\right) = 0\,. \tag{29}$$

These equations can be interpreted as the analog of Euler's equations in momentum representation (compare Eqs. (29) with Eqs. (5)) and can be rewritten in the form given by Eq. (25d).

7 CONCLUSIONS

The method of "transition to the centre and difference coordinates" allows to connect the eikonals of the optical system with the system of equations for the light ray. Four Legendre transformations which connect point, point-angle, angle-point, and point eikonals of the optical system correspond to four Legendre transformations which connect four systems of equations: Euler's equations, Hamilton's equations, and two unknown pairs of equations. One of these previously unknown pairs of equations is interpreted as Euler's equations in momentum representation.

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